Group actions on C*-algebras of a vector bundle

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Outline

- We recall the definition of the Cuntz-Pimsner algebra $\mathcal{O}_E$ of a Hermitian vector bundle $E \to X$ and discuss some examples using results from K-theory.
- We review the structure of G-vector bundles for G a compact group.
- If G acts on $E \to X$, then it acts on the C^*-correspondence $\Gamma(E)$ over $C(X)$ and on the C^*-algebra $\mathcal{O}_E$, so we can study $\mathcal{O}_E \rtimes G$.
- If the action is free and rank $E = n$, then $\mathcal{O}_E \rtimes G$ is Morita equivalent to a field of Cuntz algebras $\mathcal{O}_n$ over the orbit space X/G.
- If the action is fiberwise, then $\mathcal{O}_E \rtimes G$ becomes a continuous field of crossed products $\mathcal{O}_n \rtimes G$.
- For transitive actions, we show that $\mathcal{O}_E \rtimes G$ is Morita equivalent to a graph C^*-algebra.

Cuntz-Pimsner algebras of vector bundles

- Let $E \to X$ be a complex vector bundle with a Hermitian metric, where X is compact, metrizable and path connected.
- The set $\Gamma(E)$ of continuous sections $\xi : X \to E$ becomes a C^*-correspondence over $C(X)$, with left and right multiplications

$$(f\xi)(x) = (\xi f)(x) = f(x)\xi(x)$$

  and inner product

$$\langle \xi, \eta \rangle(x) = \langle \xi(x), \eta(x) \rangle_{E_x}.$$

- We denote by $\mathcal{O}_E$ the Cuntz-Pimsner algebra $\mathcal{O}_A(\mathcal{H})$ of the C^*-correspondence $\mathcal{H} = \Gamma(E)$ over $A = C(X)$.
- In general, $\mathcal{O}_A(\mathcal{H})$ is universal for covariant representations $\pi : A \to C, \tau : \mathcal{H} \to C$ in a C^*-algebra C, where

$$\tau(a\xi) = \pi(a)\tau(\xi), \quad \pi(\langle \xi, \eta \rangle) = \tau(\xi)^* \tau(\eta)$$

$$\pi(a) = \psi(\phi(a)) \;\text{ for }\; a \in J_{\mathcal{H}} = \phi^{-1}(\mathcal{K}_A(\mathcal{H})) \cap (\ker \phi)^{\perp}.$$

- Here $\psi : \mathcal{K}_A(\mathcal{H}) \to C, \psi(\theta_{\xi,\eta}) = \tau(\xi)\tau(\eta)^*$ and $\phi : A \to \mathcal{L}(\mathcal{H})$.

Cuntz-Pimsner algebras of vector bundles

- **Theorem** (Vasselli). If rank $E = n \geq 2$, then $\mathcal{O}_E$ is a locally trivial continuous field of Cuntz algebras $\mathcal{O}_n$.
- $\mathcal{O}_E$ is generated by $C(X)$ and $S_1, ..., S_N$ such that

$$f\,S_j = S_j\,f,\ \ S_j^* S_k = P_{jk},\ \sum_{j=1}^{N} S_j S_j^* = 1$$

  where $f \in C(X)$ and $P \in M_N \otimes C(X)$ gives E by the Serre-Swan Theorem.
- If E is a line bundle, then $\mathcal{O}_E$ is commutative with spectrum homeomorphic to the circle bundle of E.
- If E, F are line bundles over X, then $\mathcal{O}_E \cong \mathcal{O}_F$ as $C(X)$-algebras if and only if $E \cong F$ or $E \cong \bar{F}$
- **Theorem** (Dadarlat). The principal ideal $(1 - [E])K^0(X)$ determines $\mathcal{O}_E$ up to isomorphism and an inclusion $(1 - [E])K^0(X) \subseteq (1 - [F])K^0(X)$ corresponds to an unital embedding $\mathcal{O}_E \subseteq \mathcal{O}_F$.
- If E has rank $n \geq 2$, then $\mathcal{O}_E \cong C(X) \otimes \mathcal{O}_n$ if and only if $[E] - 1$ is divisible by $n - 1$ in $K^0(X)$.

Examples

- Let $X = S^2$ and let $E = TS^2 \otimes \mathbb{C}$, which is not trivial. Nevertheless, $\mathcal{O}_E \cong C(S^2) \otimes \mathcal{O}_2$.
- For $X = S^{2k}$ and $[E] = n + mt \in K^0(S^{2k}) \cong \mathbb{Z}[t]/(t^2)$ with $n \geq 3$ and $\gcd(n-1, m) = 1$ we have

$$K_0(\mathcal{O}_E) \cong \mathbb{Z}/(n-1)^2\mathbb{Z} \neq K_0(C(S^{2k}) \otimes \mathcal{O}_n).$$

- For $X = S^{2k+1}$ and $[E] = n \in K^0(S^{2k+1}) \cong \mathbb{Z}$ we have

$$\mathcal{O}_E \otimes \mathcal{K} \cong C(S^{2k+1}) \otimes \mathcal{O}_n \otimes \mathcal{K}$$

  as graded algebras.

G-vector bundles

- Let G be a topological group. A G-vector bundle is $p : E \to X$ with a continuous G action on X and E such that p is equivariant and the maps $E_x \to E_{g \cdot x}$ are linear.
- G acts on $C(X)$ by $g \cdot f(x) = f(g^{-1}x)$ and on $\Gamma(E)$ by $g \cdot \xi(x) = g\xi(g^{-1}x)$.
- **Example**. If X is a manifold and G acts smoothly on X, then $E = TX \otimes \mathbb{C}$ becomes a G-vector bundle.
- If E is a vector bundle on X, then $E^{\otimes k}$ becomes an S_k-vector bundle on X, where S_k permutes the factors, and X has a trivial action.
- If X is a point, then a G-vector bundle is just a finite dimensional representation of G.
- If X is a trivial G-space, then a G-vector bundle is a continuous family of representations E_x of G.

G-vector bundles results

- G-vector bundles E over a free G-space X correspond bijectively to vector bundles over X/G with trivial action.
- For G compact, let $\{V_i\}_{i\geq 1}$ be the set of irreducible representations.
- If X is a trivial G-space, then every G-bundle E over X is isomorphic to a direct sum $\bigoplus_i W_i \otimes E_i$, where $W_i = X \times V_i$ has the action $g \cdot (x, v) = (x, g \cdot v)$, and $E_i = Hom_G(W_i, E) \cong Hom(W_i, E)^G$ are vector bundles with trivial action.
- Any G-vector bundle E over the homogeneous space G/H is of the form $G \times_H W$ for some H-module W.
- Here $G \times_H W$ is the quotient of $G \times W$ under the action $h \cdot (g, w) = (gh^{-1}, h \cdot w)$ and G acts by $g \cdot (g', w) = (gg', w)$.

Crossed products of C^*-correspondences

- A group G acts on a C^*-correspondence $(A, \mathcal{H})$ by (α, β) if

$$\langle \beta_g(\xi), \beta_g(\eta) \rangle = \alpha_g(\langle \xi, \eta \rangle),\ \beta_g(\xi a) = \beta_g(\xi)\alpha_g(a),\ \beta_g(a\xi) = \alpha_g(a)\beta_g(\xi).$$

- For $a \in C_c(G, A), \xi \in C_c(G, \mathcal{H})$ define

$$(a\xi)(s) = \int_G a(t)\beta_t(\xi(t^{-1}s))dt,\ (\xi a)(s) = \int_G \xi(t)\alpha_t(a(t^{-1}s))dt,$$

$$\langle \xi, \eta \rangle(s) = \int_G \alpha_{t^{-1}}(\langle \xi(t), \eta(ts) \rangle)dt.$$

- The completion gives a crossed product C^*-correspondence $(A \rtimes_\alpha G, \mathcal{H} \rtimes_\beta G)$.
- For G a compact group, $A \rtimes_\alpha G$ can be identified with a subalgebra of $A \otimes \mathcal{K}(L^2(G))$ and $\mathcal{H} \rtimes_\beta G$ with a subspace of $\mathcal{H} \otimes \mathcal{K}(L^2(G))$.

Morita equivalence

- Given C^*-correspondences $\mathcal{H}$ over A and $\mathcal{M}$ over B, we say that $\mathcal{H}$ and $\mathcal{M}$ are Morita equivalent in case A and B are Morita equivalent via an imprimitivity bimodule $\mathcal{Z}$ such that $\mathcal{Z} \otimes_B \mathcal{M}$ and $\mathcal{H} \otimes_A \mathcal{Z}$ are isomorphic as C^*-correspondences from A to B.
- Using linking algebras, Muhly and Solel proved that for faithful and essential Morita equivalent C^*-correspondences $\mathcal{H}$ and $\mathcal{M}$, the Cuntz-Pimsner algebras $\mathcal{O}_A(\mathcal{H})$ and $\mathcal{O}_B(\mathcal{M})$ are Morita equivalent.
- **Theorem**. Suppose that a localy compact amenable group G acts on faithful and essential Morita equivalent C^*-correspondeces $\mathcal{H}$ and $\mathcal{M}$ over A and B respectively, via an imprimitivity bimodule $\mathcal{Z}$.
- Then $\mathcal{Z} \rtimes G$ becomes an imprimitivity bimodule between $A \rtimes G$ and $B \rtimes G$. Moreover, $\mathcal{O}_A(\mathcal{H}) \rtimes G$ is Morita equivalent to $\mathcal{O}_B(\mathcal{M}) \rtimes G$.

Results

- **Theorem** (Hao-Ng). Let G act on $(A, \mathcal{H})$. By the universal property of $\mathcal{O}_A(\mathcal{H})$ we get $\gamma : G \to \mathrm{Aut}\, \mathcal{O}_A(\mathcal{H})$.
- If G is amenable, then

$$\mathcal{O}_A(\mathcal{H}) \rtimes_\gamma G \cong \mathcal{O}_{A \rtimes_\alpha G}(\mathcal{H} \rtimes_\beta G).$$

- **Corollary**. If G compact acts on a Hermitian vector bundle $E \to X$ by isometries, then G acts on $(C(X), \Gamma(E))$ and

$$\mathcal{O}_E \rtimes G \cong \mathcal{O}_{C(X) \rtimes G}(\Gamma(E) \rtimes G).$$

- It is useful to understand the finitely generated projective module $\Gamma(E) \rtimes G$ as a kind of noncommutative bundle over $C(X) \rtimes G$, which in some cases is Morita equivalent to an abelian C^*-algebra.

Results

- **Theorem 1** (Free action). If G compact acts freely on the Hermitian vector bundle $E \to X$, then $\mathcal{O}_E \rtimes G$ is Morita equivalent with a continuous field of Cuntz algebras over X/G.
- **Example**. The group $\mathbb{Z}_2 = \{e, g\}$ acts on S^2 by $g \cdot x = -x$ and on $E = TS^2 \otimes \mathbb{C}$ by its differential dg. Since the action is free, $E/\mathbb{Z}_2$ is a vector bundle over $S^2/\mathbb{Z}_2 = \mathbb{R}P^2$.
- Moreover, $C(S^2) \rtimes \mathbb{Z}_2$ is Morita equivalent with $C(\mathbb{R}P^2)$ and it follows that $\mathcal{O}_E \rtimes \mathbb{Z}_2$ is Morita equivalent with $C(\mathbb{R}P^2) \otimes \mathcal{O}_2$.
- **Theorem 2** (Fiberwise action). If G compact acts on $E \to X$ of rank n and the action on X is trivial, then $\mathcal{O}_E \rtimes G$ is a continuous field with fibers $\mathcal{O}_n \rtimes G$.
- For $G = S_n$ we know that $\mathcal{O}_n \rtimes S_n$ is simple and purely infinite.
- If X is finite dimensional, Dadarlat gives a complete list of the UCT Kirchberg algebras D with finitely generated K-theory for which every unital separable continuous field over X with fibers isomorphic to D is automatically locally trivial or trivial.

Results

- **Theorem 3** (Transitive action). Let G be a compact group and let H be a closed subgroup. Given a Hermitian vector bundle E over $X = G/H$ we know that $E \cong G \times_H V$ for an H-module V.
- Then $\mathcal{O}_E \rtimes G$ is Morita equivalent to a graph C^*-algebra.
- Indeed, $C(G/H) \rtimes G$ is Morita equivalent with $C^*(H)$ which is a direct sum of matrix algebras.
- This in turn is Morita equivalent to $C_0(Y)$ with Y at most countable.
- Now it is known that a C^*-correspondence over $C_0(Y)$ gives rise to a discrete graph.

Selected references

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