Purely infinite dynamical systems and their C^*-algebras

AMS Special Session on C^*-algebras, Dynamical systems and Applications

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Pure infiniteness of C^*-algebras

Let A be a C^*-algebra.

1. We write $M_\infty(A)_+ = \bigcup_{n=1}^{\infty} M_n(A)_+$. Let a, b be two positive elements in $M_n(A)_+$ and $M_m(A)_+$, respectively. Write $a \precsim b$ if there exists a sequence (r_n) in $M_{m,n}(A)$ with $r_n^* b r_n \to a$.
2. A non-zero positive element a in A is said to be *properly infinite* if $a \oplus a \precsim a$. Then A is said to be *purely infinite* if there are no characters on A and if, for every pair of positive elements $a, b \in A$ such that b belongs to the closed ideal in A generated by a, one has $b \precsim a$.
3. Kirchberg and Rørdam showed that A is purely infinite if and only if every non-zero positive element a in A is properly infinite.

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Finiteness against strongly purely infiniteness of C^*-algebras

1. Kirchberg and Rørdam also introduced a stronger notion called *strongly pure infiniteness*. If A is nuclear and separable then A is strongly purely infinite if and only if $A \otimes \mathcal{O}_\infty \simeq A$.
2. Rørdam and Pasnicu then showed that pure infiniteness is equivalent to strongly pure infiniteness if A has the ideal property (IP).
3. we say A is *finite* if $1_{\tilde{A}}$ is a finite projection in $\tilde{A}$. If $M_n(A)$ are finite for all $n \in \mathbb{N}$ then we say A is *stably finite*.

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Dynamical systems

- Throughout G denotes a countable infinite discrete group, X a locally compact metrizable topological space and $\alpha : G \curvearrowright X$ a continuous action of G on X.
- $\alpha : G \curvearrowright X$ is amenable iff $C_0(X) \rtimes_r G$ is nuclear iff the transformation groupoid $X \rtimes G$ is amenable. In this case, $C_0(X) \rtimes_r G$ satisfies the UCT by a theorem of Tu.
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Comparison of open sets in Dynamical systems

Definition (Kerr, 2017)

Let F be a compact set in X and O an open set in X. We write $F \prec O$ if there exists a finite collection $\mathcal{U} = \{U_1, \ldots, U_n\}$ of open sets in X and group elements $\{s_1, \ldots, s_n\}$ such that $F \subset \bigcup_{i=1}^n U_i$ and $\bigsqcup_{i=1}^n s_i U_i \subset O$. In addition, for open sets U, V, we write $U \prec V$ if $F \prec V$ holds whenever F is a compact subset of U.

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The action $\alpha : G \curvearrowright X$ is said to have dynamical comparison if $U \prec V$ for every non-empty open sets $U, V \subset X$ satisfying $\mu(U) < \mu(V)$ for all $\mu \in M_G(X)$, where $M_G(X)$ is the set consisting of all G-invariant probability Borel measure.

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Purely infinite dynamical systems

Definition (X. Ma)

Let $\alpha : G \curvearrowright X$. we write $U \prec_d V$ if for any compact set $F \subset U$ there are disjoint non-empty open sets $O_1, O_2 \subset V$ such that $F \prec O_1$ and $F \prec O_2$.

Definition (M.)

Let $\alpha : G \curvearrowright X$. We say the action α

1. *is purely infinite if $U \prec_d V$ whenever $U \subset G \cdot V$ for any open sets U, V in X.*
2. *has paradoxical comparison if $U \prec_d U$ for any open set U in X.*
3. *is weakly purely infinite if $U \prec V$ whenever $U \subset G \cdot V$ for any open sets U, V in X*

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Relations among these three notions

1. In general one easily has (1)$\Rightarrow$(2) and (1) $\Rightarrow$(3).
2. If the action α is minimal then all of these three notions are equivalent to dynamical comparison in the case $M_G(X)=\emptyset$, i.e., $U \prec V$ for any open sets U, V in X.
3. If α is not minimal then one actually could establish (1)$\Leftrightarrow$(2). However, (3) is strictly weaker than (1) and (2) because the trivial action of the group G on X is weakly purely infinite but is not purely infinite.
4. Nevertheless, if the space X is zero-dimensional and the action has no global fixed points (i.e. $G\cdot\{x\}=\{x\}$), then one can show (3) is equivalent to (1) and thus also (2).

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Known examples

- Strong boundary actions (on a Compact space X) defined by Laca-Spielberg is an example. E.g. $\mathbb{F}_2$ acting on its boundary. Here strong boundary action means for any two non-empty open sets U_1, U_2 there are group elements $g_1, g_2 \in G$ such that $g_1 U_1 \cup g_2 U_2 = X$.
- A generalization, called the n-filling actions, by Jolissaint-Robertson are also examples of purely infinite actions, in which n-filling means for any n non-empty open sets $U_1, \ldots, U_n$ there are group elements $g_1, \ldots, g_n \in G$ such that $\bigcup_{i=1}^{n} g_i U_i = X$.
- Suppose α is minimal and there is a group element $g \in G$ having a fixed point x_0 which is an attractor in the sense that there is an open neighborhood W of x_0 such that $\{g^n(W) : n \in \mathbb{N}\}$ form a neighborhood basis at x_0. Then α is purely infinite. This covers some examples of local boundary actions defined by Laca-Spielbeg

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Permanence properties I

- Pure infiniteness of actions is not preserved by extensions. Here is an example due to Hanfeng Li. Start with a Cantor pure infinite system $\alpha : G \curvearrowright X$. Let G^* denote the one-point compactification $G \cup \{\infty\}$ for G. Note $\beta : G \curvearrowright G^*$ is the natural action given by $g \cdot h = gh$ for $h \in G$ and $g \cdot \infty = \infty$. Define $\gamma : G \curvearrowright X \times G^*$ by $\gamma_g(x, p) = (\alpha_g(x), \beta_g(p))$. Then γ is an extension of α. Consider the open set $X \times \{h\}$ for $h \in G$. It is not hard to see it is impossible to find two disjoint non-empty open subsets $V_1 \times \{h\}$ and $V_2 \times \{h\}$ of $X \times \{h\}$. such that $X \times \{h\} \prec V_i \times \{h\}$ for $i = 1, 2$.
- Since α has no G-invariant probability measure on X, neither does γ because γ is an extension. Nevertheless, one can show for any open set of the form $O \times \{h\}$ in $X \times G$ there is a infinite Borel regular G-invariant measure μ such that $\mu(O \times \{h\}) = 1$. Therefore Hanfeng' s example in fact has a flavor of finiteness.

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Permanence properties II

- Natural question: Is there a Cantor dynamical system having no non-trivial Borel invariant measure and also not purely infinite.
- Pure infiniteness of actions is preserved by inverse limits of dynamical systems.
- Suzuki constructed a class of group actions by inverse limits of $\mathbb{F}_n$ acting on its boundary when he studied the K-theory of Kirchberg algebras. Therefore his examples of dynamical systems are in fact purely infinite.
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A dynamical model for $\mathcal{O}_2$ I

Kumjian and Archbold independently observe that the Cuntz algebra $\mathcal{O}_2$ has a dynamical model in the sense that there is an action α_0 of $\mathbb{Z}_2 * \mathbb{Z}_3$ on the Cantor set X such that $\mathcal{O}_2 \simeq C(X) \rtimes_r (\mathbb{Z}_2 * \mathbb{Z}_3)$ where α_0 is defined as follows. Identify X by $\{0,1\}^{\mathbb{N}}$ and let φ and ψ be two homeomorphism on X given by

$$(0, x_2, x_3, \dots) \xrightarrow{\varphi} (1, x_2, x_3, \dots) \xrightarrow{\varphi} (0, x_2, x_3, \dots)$$

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$$(0, x_2, x_3, \dots) \xrightarrow{\psi} (1, 1, x_2, x_3, \dots) \xrightarrow{\psi} (1, 0, x_2, x_3, \dots) \xrightarrow{\psi} (0, x_2, x_3, \dots)$$

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- Now $X = N_0 \sqcup N_1$. In addition, choose two disjoint open sets $N_{z_1 z_2,\ldots,z_n}$ and $N_{y_1 y_2,\ldots,y_m} \subset O$. Without loss of generality, one can assume $n, m \geq 2$. Now, it suffices to show that there are $g_1, g_2 \in \mathbb{Z}_2 * \mathbb{Z}_3$ such that $g_1 N_0 = N_{z_1 z_2,\ldots,z_n}$ and $g_2 N_1 = N_{y_1 y_2,\ldots,y_m}$.
- For $N_{z_1 z_2,\ldots,z_n}$, where $n \geq 2$, one has
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- This implies that there is a $g \in \mathbb{Z}_2 * \mathbb{Z}_3$ such that $gN_{z_1 z_2,\dots,z_n} = N_{z_2,\dots,z_n}$. Indeed,
  1. If $z_1 = z_2 = 1$ define $g = \varphi \circ \psi^{-1}$.
  2. If $z_1 = 1$ and $z_2 = 0$ then define $g = \psi$.
  3. If $z_1 = 0$, by third condition above, one can always reduce the problem to the case $z_1 = 1$ above.
- By induction there is an $h \in \mathbb{Z}_2 * \mathbb{Z}_3$ such that $hN_{z_1 z_2,\dots,z_n} = N_{z_n}$. If $z_n = 0$ we are done and if $z_n = 1$ then $\varphi(hN_{z_1 z_2,\dots,z_n}) = N_0$.
- Thus there is a $g_1 \in \mathbb{Z}_2 * \mathbb{Z}_3$ such that $g_1 N_0 = N_{z_1 z_2,\dots,z_n}$. The same method shows that there is an h_2 such that $h_2 N_0 = N_{y_1 y_2,\dots,y_m}$. Then define $g_2 = h_2 \circ \varphi$. Then $g_2 N_1 = N_{y_1 y_2,\dots,y_m}$.
- In fact, this action is a strong boundary action.

A dynamical model for $\mathcal{O}_2$ III

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  1. If $z_1 = z_2 = 1$ define $g = \varphi \circ \psi^{-1}$.
  2. If $z_1 = 1$ and $z_2 = 0$ then define $g = \psi$.
  3. If $z_1 = 0$, by third condition above, one can always reduce the problem to the case $z_1 = 1$ above.
- By induction there is an $h \in \mathbb{Z}_2 * \mathbb{Z}_3$ such that $hN_{z_1 z_2,\dots,z_n} = N_{z_n}$. If $z_n = 0$ we are done and if $z_n = 1$ then $\varphi(hN_{z_1 z_2,\dots,z_n}) = N_0$.
- Thus there is a $g_1 \in \mathbb{Z}_2 * \mathbb{Z}_3$ such that $g_1 N_0 = N_{z_1 z_2,\dots,z_n}$. The same method shows that there is an h_2 such that $h_2 N_0 = N_{y_1 y_2,\dots,y_m}$. Then define $g_2 = h_2 \circ \varphi$. Then $g_2 N_1 = N_{y_1 y_2,\dots,y_m}$.
- In fact, this action is a strong boundary action.

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2. In the case α is amenable, Sierakowski showed that $C_0(X)$ separates ideals of $C_0(X) \rtimes_r G$ if and only if α is essentially free.

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Let $\alpha : G \curvearrowright X$. Suppose $C_0(X)$ separates ideals of $C_0(X) \rtimes_r G$ and there are only finitely many G-invariant closed sets in X. If α is purely infinite then $C_0(X) \rtimes_r G$ is strongly purely infinite.

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C^*-algebras arising from purely infinite dynamical systems

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Corollary

Let $\alpha : G \curvearrowright X$ be a minimal topologically free action. Suppose that the action α is purely infinite Then the reduced crossed product $C_0(X) \rtimes_r G$ is strongly purely infinite. If α is also amenable then $C_0(X) \rtimes_r G$ is a Kirchberg algebra.

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Let $\alpha : G \curvearrowright X$ be a minimal topologically free continuous action of G on X. Suppose that the action α has dynamical comparison. Then the reduced crossed product $C_0(X) \rtimes_r G$ is simple and is either stably finite or strongly purely infinite.

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Let $\alpha : G \curvearrowright X$ be a minimal action. Let Y be a locally compact Hausdorff space and $\beta : G \curvearrowright X \times Y$ by $\beta_g(x, y) = (\alpha_g(x), y)$. Suppose α is purely infinite then so is β.

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There exists a non-simple strongly purely infinite C^*-algebra A, for example, $\mathcal{O}_2 \otimes C_0(\mathbb{R})$, which has a purely infinite dynamical model. However, it has no dynamical model implemented by an n-filling or locally boundary action.

Dynamical models of non-simple purely infinite C^*-algebras

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The end

Thank you!