Free Stein Irregularity and Dimension

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January 16th, 2020

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- (M, τ) a von Neumann algebra equipped with a faithful normal tracial state
- $X := (x_1, \ldots, x_n) \in M^n$ such that $X^* = X$ and $M = W^*(X)$
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$$i = 1, \ldots, n \qquad \partial_i \colon \mathbb{C}\langle X \rangle \to \mathbb{C}\langle X \rangle \otimes \mathbb{C}\langle X \rangle^\circ$$

$$\partial_i(x_{j_1} \cdots x_{j_d}) := \sum_{k=1}^{d} \delta_{i=j_k} x_{j_1} \cdots x_{j_{k-1}} \otimes x_{j_{k+1}} \cdots x_{j_d}$$

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Warning

If $x_1, \ldots, x_n$ admit algebraic relations, these need not be well-defined operators.

Non-commutative gradient:

$$\partial \colon \mathbb{C}\langle X\rangle \to (\mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle^{\circ})^n$$
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$$\mathscr{J} : \mathbb{C}\langle X\rangle^n \to M_n(\mathbb{C}\langle X\rangle \otimes \mathbb{C}\langle X\rangle^{\circ})$$
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$$\mathscr{J}(X) = \begin{pmatrix} 1\otimes 1 & & 0 \\ & \ddots & \\ 0 & & 1\otimes 1 \end{pmatrix} =: \mathbb{1}$$

- Want to view ∂ and $\mathscr{J}$ as a densely defined (unbounded) operators:

$$\partial \colon L^2(M) \to L^2(M \bar{\otimes} M^\circ)^n \qquad \mathscr{J} \colon L^2(M)^n \to M_n(L^2(M \bar{\otimes} M^\circ))$$

with adjoints:

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$$\int g_\epsilon(t) p(t)\, d\mu(t) = \iint A_\epsilon(s,t) \frac{p(s) - p(t)}{s - t}\, d\mu(s) d\mu(t) \qquad \forall p \in \mathbb{C}[x].$$

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Then

$$\Sigma^*(x)^2 \leq \|A_\epsilon - \mathbb{1}\|_2^2 = \iint |A_\epsilon(s,t) - 1|^2\; d\mu(s) d\mu(t) \to \iint 1_{s=t}(s,t)\; d\mu(s) d\mu(t).$$

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Definition [Voiculescu; 1998]

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In particular, $\boxed{\Sigma^*(X) = 0} \Rightarrow$ no algebraic relations and ∂, $\mathscr{J}$ are well-defined operators.

Lemma [Voiculescu, 1998]

$$ {}_{\mathbb{C}\langle X\rangle\otimes\mathbb{C}\langle X\rangle^\circ}\mathrm{dom}\,(\partial^*) < {}_{\mathbb{C}\langle X\rangle\otimes\mathbb{C}\langle X\rangle^\circ} L^2(M\bar{\otimes}M^\circ)^n $$

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2. Let X be an n-tuple generating the following finite-dimensional algebra:

$$(M, \tau) = \bigoplus_{i=1}^{d} (M_{k_i}(\mathbb{C}), \alpha_i \frac{1}{k_i} \mathrm{Tr}).$$

Then

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- With $M = W^*(X)$, consider

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$$\sigma(X) = \sigma(Y)$$

for any $Y \in \mathbb{C}\langle X\rangle^m$ satisfying $\mathbb{C}\langle Y\rangle = \mathbb{C}\langle X\rangle$.

- With $M = W^*(X)$, consider

$$\mathrm{Der}_{1\otimes 1}(\mathbb{C}\langle X\rangle) := \{d \colon \mathbb{C}\langle X\rangle \to L^2(M\bar{\otimes}M^\circ) \mid d \text{ is a derivation with } 1\otimes 1 \in \mathrm{dom}\,(d^*)\}$$

- $$L^2(M\bar{\otimes}M^\circ)_{\mathbb{C}\langle X\rangle\otimes\mathbb{C}\langle X\rangle^\circ} \rightsquigarrow \mathrm{Der}_{1\otimes 1}(\mathbb{C}\langle X\rangle)_{\mathbb{C}\langle X\rangle\otimes\mathbb{C}\langle X\rangle^\circ}$$

Theorem [Charlesworth+N., 2019]

There is a 1-1 correspondence

$$\begin{aligned}\mathrm{Der}_{1\otimes 1}(\mathbb{C}\langle X\rangle) &\leftrightarrow \mathrm{dom}\,(\partial^*)\\ d &\mapsto (Jd(x_1),\cdots,Jd(x_n))\end{aligned}$$

which intertwines the right and left actions of $\mathbb{C}\langle X\rangle\otimes\mathbb{C}\langle X\rangle^\circ$. Consequently,

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Remark

First non-microstates quantity known to be an algebra invariant.

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Question

If $W^*(X) = R$ the hyperfinite II_1 factor, is $\sigma(X) = 1$?