STABLE RANK OF $\mathrm{C}(X) \rtimes \Gamma$

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Dedicated to Professor George A. Elliott on the occasion of his 75th birthday

Abstract. It is shown that, for an arbitrary free and minimal $\mathbb{Z}^n$-action on a compact Hausdorff space X, the crossed product C*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}^n$ always has stable rank one, i.e., invertible elements are dense. This generalizes a result of Alboiu and Lutley on $\mathbb{Z}$-actions.

In fact, for any free and minimal topological dynamical system (X, Γ), where Γ is a countable discrete amenable group, if it has the uniform Rokhlin property and Cuntz comparison of open sets, then the crossed product C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ has stable rank one. Moreover, in this case, the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ absorbs the Jiang-Su algebra tensorially if, and only if, it has strict comparison of positive elements.

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1. Introduction

The topological stable rank of a unital C*-algebra A, denoted by $\mathrm{tsr}(A)$, is introduced by Rieffel in his seminal paper [21] as a topological version of the Bass stable rank of a ring: Denote by

$$Lg_n = \{(x_1, x_2, ..., x_n) \in A^n : Ax_1 + Ax_2 + \cdots + Ax_n = A\}.$$

Then the topological stable rank of A, denote by $\mathrm{tsr}(A)$, is the smallest n such that Lg_n is dense in A^n (if no such n exists, then the topological stable rank of A is ∞). It was shown in

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[12] that the topological stable rank of a C*-algebra agrees with its Bass stable rank. Thus, we may just refer it as stable rank.

The stable rank models dimension of a topological space: Consider the commutative C*-algebra $\mathrm{C}(X)$, where X is a compact Hausdorff space. Its stable rank is $\lfloor \frac{\dim(X)}{2} \rfloor + 1$, where $\lfloor \cdot \rfloor$ denotes the integer part. It is also shown in [30] that for any $n \in \{1, 2, ..., \infty\}$, there exists a simple unital separable C*-algebra A (A actually can be chosen to be the limit of an inductive sequence of homogeneous C*-algebras) such that $\mathrm{tsr}(A) = n$.

The class of C*-algebras with stable rank one is particularly interesting. Any such C*-algebra A is stably finite, has cancellation of projections, and has the property that $\mathrm{K}_1(A)$ is canonically isomorphic to $\mathrm{U}(A)/\mathrm{U}_0(A)$. It is also well known that a C*-algebra has stable rank one if, and only if, $A = \overline{\mathrm{GL}(A)}$, where $\mathrm{GL}(A)$ denotes the group of invertible elements of A.

Many classes of simple C*-algebras have been shown to have stable rank one. For instance, any simple unital finite C*-algebra which absorbs a UHF algebra or, in general, absorbs the Jiang-Su algebra $\mathcal{Z}$, has stable rank one (see [22] and [24], respectively). These C*-algebras certainly are well behaved from the perspective of the classification program.

On the other hand, even beyond the classifiable C*-algebras, remarkably, it was shown by Elliott, Ho, and Toms ([9], also see [13]) that any simple unital AH algebra with diagonal maps (this class of C*-algebras contains the exotic AH algebras of [29] and [28], which cannot be classified by the ordered K-groups together with the traces—the Elliott invariant), no matter classifiable or not, always has stable rank one. This result actually is the main motivation of the current paper: Consider the C*-algebra of a minimal homeomorphism. In general, its behaviors are expected to be parallel to the behaviors of AH algebras with diagonal maps (see, for instance, [15] and [10] on the classifiability and mean dimension), and it had been speculated for a while in the C*-algebra community that the C*-algebra of a minimal homeomorphism should always has stable rank one. (Note that, as shown in [11], there exists a minimal homeomorphism of an infinite compact Hausdorff space such that the corresponding C*-algebra is not classifiable by its Elliott invariant.)

A considerable amount of work has been done concerning this question, and finally it is solved recently by Alboiu and Lutley in [1]. For the C*-algebra of a minimal homeomorphism, one can consider the orbit-breaking subalgebra, which was introduced by Putnam for Cantor dynamical systems ([20]) and then constructed for a general minimal homeomorphism by Q. Lin ([14]). It was shown by Archey and Phillips ([2]) that if the orbit-breaking subalgebra has stable rank one, then the transformation group C*-algebra must have stable rank one. If the minimal dynamical system has a Cantor factor (so that the orbit-breaking subalgebra is an AH algebra with diagonal maps), with the result of [9], one has that the transformation group C*-algebra has stable rank one (see [2]) (this result is also generalized by Suzuki in [26] to the C*-algebra of a minimal almost finite groupoid). For a general minimal homeomorphism ($\mathbb{Z}$-action), the orbit-breaking subalgebra might not be AH, but rather a unital inductive limit of subhomogeneous C*-algebras with diagonal maps (the DSH algebras of [1]). Alboiu and Lutley show in [1] that any unital simple DSH algebra has stable rank one, and thus the C*-algebra of a minimal homeomorphism has stable rank one.

Beyond the case of $\mathbb{Z}$-actions, however, it is not clear how to construct orbit-breaking subalgebras in general. So, instead, one may consider the Uniform Rokhlin Property (URP) and Cuntz-comparison of Open Sets (COS) (see Definitions 2.19 and 2.21) for a topological dynamical system (X, Γ), where Γ is a countable discrete amenable group. These two properties are introduced in [17], and it is shown that, under the assumption of the (URP) and (COS), the radius of comparison of the crossed product C*-algebra $C(X) \rtimes \Gamma$ is dominated by half of the mean dimension of (X, Γ) ([17]), and the C*-algebra $C(X) \rtimes \Gamma$ is classified by its Elliott invariant if (X, Γ) has mean dimension zero ([18]). Moreover, any free and minimal $\mathbb{Z}^d$-action have the (URP) and (COS) ([16]).

In this paper, we still consider these two properties, and we show that if a free and minimal Γ-action has the (URP) and (COS), then the transformation group C*-algebra must have stable rank one (Theorem 7.8). Since any free and minimal $\mathbb{Z}^d$-actions have the (URP) and (COS), the C*-algebra $C(X) \rtimes \mathbb{Z}^d$ has stable rank one, no matter it is classifiable or not:

Theorem (Corollary 7.9). *Let $\mathbb{Z}^d$ act freely and minimally on a compact Hausdorff space X. Then* $\mathrm{tsr}(C(X) \rtimes \mathbb{Z}^d) = 1$.

As consequences of stable rank one, we obtain the following properties of the crossed product C*-algebra $A = C(X) \rtimes \mathbb{Z}^d$, where $(X, \mathbb{Z}^d)$ is free and minimal:

- A has cancellation of projections, cancellation in Cuntz semigroup, and $\mathrm{U}(A)/\mathrm{U}_0(A) \cong \mathrm{K}_1(A)$ (Corollary 7.11).
- Approximately unitary equivalence classes of homomorphisms from an AI algebra to A is determined by the induced maps on Cuntz semigroups (Corollary 7.12).
- Any strictly positive lower semicontinous affine function on $\mathrm{T}(A)$ can be realized as the rank function of some positive element of $A \otimes \mathcal{K}$ (Corollary 7.13).
- A absorbs the Jiang-Su algebra tensorially if, and only if, A has strict comparison of positive elements (Corollary 7.14). That is, A satisfies the Toms-Winter conjecture.
- The real rank of A is either 0 or 1 (Corollary 7.15).

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2. Notation and preliminaries

2.1. **Topological Dynamical Systems.**

Definition 2.1. Consider a topological dynamical system (X, Γ), where X is a separable compact Hausdorff space, Γ is a discrete group which acts on X from the right. The dynamical system (X, Γ) is said to be minimal if

$$Y\gamma = Y, \quad \gamma \in \Gamma$$

for some closed set $Y \subseteq X$ implies $Y = \varnothing$ or $Y = X$; and the dynamical system (X, Γ) is said to be free if

$$x\gamma = x$$

for some $x \in X$ and $\gamma \in \Gamma$ implies $\gamma = e$.

Definition 2.2. A Borel measure μ on X is invariant under the action σ if for any Borel set $E \subseteq X$, one has

$$\mu(E) = \mu(E\gamma), \quad \gamma \in \Gamma.$$

Denote by $\mathcal{M}_1(X, \Gamma)$ the set of all invariant Borel probability measures on X. It is a Choquet simplex under the weak* topology.

Definition 2.3. Let Γ be a countable discrete group. Let $K \subseteq \Gamma$ be a finite set and let $\delta > 0$. Then a finite set $E \subseteq \Gamma$ is said to be (K, ε)-invariant if

$$\frac{|EK \Delta E|}{|E|} < \varepsilon.$$

The group Γ is amenable if there is a sequence (Γ_n) of finite subsets of Γ such that for any (K, ε), there is N such that Γ_n is (K, ε)-invariant for any $n > N$. The sequence (Γ_n) is called a Følner sequence.

The K-interior of a finite set $E \subseteq \Gamma$ is defined as

$$\mathrm{int}_K(E) = \{\gamma \in E : \gamma K \subseteq E\},$$

and the K-boundary of E is defined as

$$\partial_K E := E \setminus \mathrm{int}_K(E) = \{\gamma \in E : \gamma\gamma' \notin E \text{ for some } \gamma' \in K\}.$$

Note that

$$|E \setminus \mathrm{int}_K(E)| \leq |K| \, |EK \setminus E| \leq |K| \, |EK \Delta E| \, ,$$

and hence for any $\varepsilon > 0$, if E is $(K, \frac{\varepsilon}{|K|})$-invariant, then

$$\frac{|E \setminus \mathrm{int}_K(E)|}{|E|} < \varepsilon.$$

Remark 2.4. If a set $E \subseteq \Gamma$ is $(\mathcal{F}, \varepsilon)$-invariant, then, for any $\gamma \in \Gamma$, the left translation γE is again $(\mathcal{F}, \varepsilon)$-invariant.

Definition 2.5. An (exact) tiling of a discrete group consists of

- a finite collection $\mathcal{S} = \{\Gamma_1, ..., \Gamma_n\}$ of finite subsets of Γ containing the unit e, called the shapes,
- a finite collection $\mathcal{C} = \{C(S) : S \in \mathcal{S}\}$ of disjoint subsets of Γ, called center sets,

such that the left translations

$$cS, \quad c \in C(S), \ S \in \mathcal{S}$$

form a partition of Γ.

Remark 2.6. If Γ is amenable, it follows from [6] that for any finite set $\mathcal{F} \subseteq \Gamma$ and any $\varepsilon > 0$, there is a tiling of Γ such that all its shapes are $(\mathcal{F}, \varepsilon)$-invariant.

2.2. **Crossed product C*-algebras.** Consider a topological dynamical system (X, Γ). Then the group Γ acts (from the left) on the C*-algebra $\mathrm{C}(X)$ by

$$\gamma(f) = f \circ \gamma.$$

The (full) crossed product C*-algebra $A = \mathrm{C}(X) \rtimes \Gamma$ is defined to be the universal C*-algebra

$$\mathrm{C}^*\{f, u_\gamma : u_\gamma f u_\gamma^* = f \circ \gamma,\ u_{\gamma_1} u_{\gamma_2}^* = u_{\gamma_1 \gamma_2^{-1}},\ u_e = 1_A,\ f \in \mathrm{C}(X),\ \gamma, \gamma_1, \gamma_2 \in \Gamma\}.$$

The C*-algebra A is nuclear if Γ is amenable (see, for instance, Corollary 7.18 of [31]). If, moreover, σ is minimal, the C*-algebra A is simple (Theorem 5.16 of [7] and Théorème 5.15 of [33]), i.e., A has no non-trivial two-sided ideals.

2.3. **Cuntz-sub-equivalence and rank functions.**

Definition 2.7. Let A be a C*-algebra, and let $a, b \in A^+$. The element a is said to be Cuntz sub-equivalent to b, denoted by $a \precsim b$, if there are x_i, y_i, $i = 1, 2, ...$, such that

$$\lim_{i \to \infty} x_i b y_i = a.$$

Example 2.8. Let $f, g \in \mathrm{C}(X)$ be positive elements, and consider the open sets

$$E := f^{-1}(0, +\infty) \quad \text{and} \quad F := g^{-1}(0, +\infty).$$

Then $f \precsim g$ if and only if $E \subseteq F$. That is, their Cuntz equivalence classes are determined by their open supports.

Throughout this paper, we use the following notation:

Definition 2.9. For any $\varepsilon > 0$, define the function $f_\varepsilon : [0,1] \to [0,1]$ by

$$f_\varepsilon(t) = \begin{cases} 0, & t < \varepsilon/2, \\ \text{linear}, & \varepsilon/2 \leq t < \varepsilon, \\ 1, & t \geq \varepsilon. \end{cases}$$

Lemma 2.10 (Proposition 2.4(iv) of [23])**.** *Let A be a C*-algebra, and let $a, b \in A$ be positive. If $a \precsim b$, then for any $\delta > 0$, there is $\varepsilon > 0$ and $r \in A$ such that*

$$f_\delta(a) = r^* f_\varepsilon(b) r.$$

In particular, denoted by $v = f_\varepsilon^{\frac{1}{2}}(b) r \in A$, one has

$$f_\delta(a) = v^* v \quad \text{and} \quad v v^* \in \mathrm{Her}(b).$$

Definition 2.11. Let A be a C*-algebra, let $\mathrm{T}(A)$ denote the set of all tracial states of A, equipped with the topology of pointwise convergence. Note that if A is unital, the set $\mathrm{T}(A)$ is a Choquet simplex.

Let a be a positive element of $\mathrm{M}_\infty(A)$ and $\tau \in \mathrm{T}(A)$; define

$$\mathrm{d}_\tau(a) := \lim_{n \to \infty} \tau(a^{\frac{1}{n}}) = \mu_\tau(\mathrm{sp}(a) \cap (0, +\infty)), \quad a \in A^+,$$

where μ_τ is the Borel measure induced by τ on the spectrum of a. It is well known that if $a \precsim b$, then

$$\mathrm{d}_\tau(a) \leq \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A).$$

Example 2.12. Consider $h \in \mathrm{C}(X)^+$ and let μ be a Borel probability measure on X, where X is a compact Hausdorff space. Then

$$\mathrm{d}_{\tau_\mu} = \mu(f^{-1}(0, +\infty)),$$

where τ_μ is the trace of $\mathrm{C}(X)$ defined by

$$\tau_\mu(f) = \int f \mathrm{d}\mu, \quad f \in \mathrm{C}(X).$$

If $A = \mathrm{M}_n(\mathrm{C}_0(X))$, where X is a locally compact Hausdorff space. Then, for any positive element $a \in \mathrm{M}_\infty(A) \cong \mathrm{M}_\infty(\mathrm{C}_0(X))$ and any $\tau \in \mathrm{T}(A)$, one has

$$\tau(a) = \int_X \frac{1}{n}\mathrm{Tr}(a(x))d\mu_\tau \quad \text{and} \quad \mathrm{d}_\tau(a) = \int_X \frac{1}{n}\mathrm{rank}(a(x))d\mu_\tau,$$

where μ_τ is the Borel measure on X induced by τ.

Definition 2.13. For each open set $E \subseteq X$, pick a continuous function

$$\varphi_E : X \to [0, 1] \tag{2.1}$$

such that

(1) $E = \varphi_E^{-1}((0, 1])$ and
(2) there is an open set $V \subseteq E$ such that $\varphi_E|_V = 1$. (In particular, $\|\varphi_E\| = 1$.)

For instance, one can pick $\varphi_E(x) = \min\{\frac{1}{\varepsilon}d(x, X \setminus E), 1\}$, where d is a compatible metric on X and $\varepsilon > 0$ is sufficiently small. This notation will be used throughout this paper.

Note that the hereditary sub-C*-algebra $\overline{\varphi_E A \varphi_E}$ is independent of the choice of individual function φ_E, where A is a C*-algebra containing $\mathrm{C}(X)$. Therefore, one also denotes $\overline{\varphi_E A \varphi_E}$ by $\mathrm{Her}(E)$ in the paper.

2.4. **Order zero maps and Rokhlin towers.**

Definition 2.14 (Order zero maps)**.** Let A, B be C*-algebras. A linear map $\phi : A \to B$ is said to be order zero if

$$a \perp b \Longrightarrow \phi(a) \perp \phi(b), \quad a, b \in A^+.$$

Let A be a C*-algebras and $\phi : \mathrm{M}_n(\mathbb{C}) \to A$ is a c.p. order zero map. Let $C := \mathrm{C}^*(\phi(\mathrm{M}_n)) \subseteq A$, and let $h = \phi(1_n)$. Then $h \in C \cap C'$, $\|h\| = \|\phi\|$, and there is a homomorphism $\pi_\phi : \mathrm{M}_n(\mathbb{C}) \to \mathcal{M}(C) \cap \{h\}' \subseteq A^{**}$ such that

$$\phi(a) = \pi_\phi(a)h, \quad a \in \mathrm{M}_n(\mathbb{C}).$$

Moreover,

$$C \cong \mathrm{M}_n(\mathrm{C}_0((0, 1])).$$

Definition 2.15. Let $\phi : \mathrm{M}_n(\mathbb{C}) \to A$ be a c.p. order zero map, and let $f \in \mathrm{C}_0((0, \|h\|])$, where $h = \phi(1_n)$. Then the map

$$\mathrm{M}_n(\mathbb{C}) \ni a \mapsto \pi_\phi(a)f(h) \in A$$

is again an order zero map, where π_ϕ is as above. Denote this new order zero map by $f(\phi)$.

An order zero map $\psi : \mathrm{M}_n(\mathbb{C}) \to A$ is said to be extendable if there is a c.p. order zero map $\psi' : \mathrm{M}_n(\mathbb{C}) \to A$ such that $\phi = f_\delta(\psi')$ for some $\delta > 0$.

The following is well known:

Lemma 2.16 ([32]). *Let $v_1, v_2, ..., v_n \in A$, where A is a C^*-algebra, such that*

- $v_1^*v_1 = v_2^*v_2 = \cdots = v_n^*v_n$,
- $v_1^*v_1, v_1v_1^*, v_2v_2^*, ..., v_nv_n^*$ *are mutually orthogonal, and*
- $\|v_1^*v_1\| = 1$.

Then there is an order zero map $\phi : \mathrm{M}_{n+1}(\mathbb{C}) \to A$ such that $\|\phi\| = 1$,

$$\phi(e_{0,0}) = v_1^*v_1, \quad \phi(e_{i,i}) = v_iv_i^*, \quad i = 1, 2, ..., n.$$

Definition 2.17. A Rokhlin tower of a dynamical system (X, Γ) is a pair (B, Γ_0), where $B \subseteq X$ and $\Gamma_0 \subseteq \Gamma$ is finite, such that

$$B\gamma, \quad \gamma \in \Gamma_0,$$

are mutually disjoint. It is an open tower if the base set B is open. Without loss of generality, one may assume that Γ_0 contains the unit of Γ.

One can naturally construct order-zero maps from Rokhlin towers, as follows:

Let (B, Γ_0) be a tower, and pick a positive function $e : X \to [0, 1]$ such that $e^{-1}((0, 1]) \subseteq B$. Let $\gamma_1, \gamma_2 \in \Gamma_0$ and consider

$$v := u_{\gamma_2}^* e^{\frac{1}{2}} u_{\gamma_1}.$$

Then

$$v^*v = u_{\gamma_1}^* e u_{\gamma_1} = e \circ \gamma_1^{-1} \quad \text{and} \quad vv^* = u_{\gamma_2}^* e u_{\gamma_2} = e \circ \gamma_2^{-1}.$$

In general, if $F_1, F_2 \subseteq \Gamma_0$ are two disjoint sets with $|F_1| = |F_2|$. Pick a one-to-one correspondence $\theta : F_1 \to F_2$, and consider

$$v := \sum_{\gamma \in F_1} u_{\theta(\gamma)}^* e^{\frac{1}{2}} u_\gamma.$$

Then

$$\begin{aligned} v^*v &= \sum_{\gamma_1, \gamma_2 \in F_1} u_{\gamma_1}^* e^{\frac{1}{2}} u_{\theta(\gamma_1)} u_{\theta(\gamma_2)}^* e^{\frac{1}{2}} u_{\gamma_2} \\ &= \sum_{\gamma_1, \gamma_2 \in F_1} u_{\gamma_1}^* u_{\theta(\gamma_1)} (e^{\frac{1}{2}} \circ \theta(\gamma_1)^{-1})(e^{\frac{1}{2}} \circ \theta(\gamma_2)^{-1}) u_{\theta(\gamma_2)}^* u_{\gamma_2} \\ &= \sum_{\gamma \in F_1} u_\gamma^* e u_\gamma, \end{aligned}$$

and the same calculation shows that

$$vv^* = \sum_{\gamma \in F_1} u_{\theta(\gamma)}^* e u_{\theta(\gamma)} = \sum_{\gamma \in F_2} u_\gamma^* e u_\gamma.$$

Now, suppose there are mutually disjoint sets

$$\Gamma_{0,1}, \Gamma_{0,2}, ..., \Gamma_{0,n} \subseteq \Gamma_0$$

such that

$$|\Gamma_{0,1}| = |\Gamma_{0,2}| = \cdots = |\Gamma_{0,n}|.$$

Consider

$$e_1 := \sum_{\gamma\in\Gamma_{0,1}} u_\gamma^* e u_\gamma,\ \ldots,\ e_n := \sum_{\gamma\in\Gamma_{0,n}} u_\gamma^* e u_\gamma.$$

Then the above calculation shows that there are $v_1, v_2, \ldots, v_{n-1}$ such that

$$v_1^*v_1 = v_2^*v_2 = \cdots = v_{n-1}^*v_{n-1} = e_1$$

and

$$v_1v_1^* = e_2, v_2v_2^* = e_2, \ldots, v_{n-1}v_{n-1}^* = e_n.$$

So, there is an order zero map $\phi : \mathrm{M}_n(\mathbb{C}) \to A$ such that

$$\phi(e_{i,i}) = e_i, \quad i = 1, \ldots, n.$$

In summary, one has the following lemma:

Lemma 2.18. *Let (B, Γ_0) be a Rokhlin tower, and let $\Gamma_{0,1}, \Gamma_{0,2}, \ldots, \Gamma_{0,n} \subseteq \Gamma_0$ be mutually disjoint sets such that*

$$|\Gamma_{0,1}| = |\Gamma_{0,2}| = \cdots = |\Gamma_{0,n}|.$$

Let $e : X \to [0,1]$ be a continuous function such that $e^{-1}((0,1]) \subseteq B$. Set

$$e_1 := \sum_{\gamma\in\Gamma_{0,1}} u_\gamma^* e u_\gamma,\ \ldots,\ e_n := \sum_{\gamma\in\Gamma_{0,n}} u_\gamma^* e u_\gamma.$$

Then there is an order zero map $\phi : \mathrm{M}_n(\mathbb{C}) \to A$ such that

$$\phi(e_{i,i}) = e_i, \quad i = 1, \ldots, n.$$

2.5. **Uniform Rokhlin property and Cuntz comparison of open sets.**

Definition 2.19 ([17])**.** A dynamical system (X, Γ) is said to have the uniform Rokhlin property (URP) if for any finite set $\mathcal{F}$, any $\varepsilon > 0$, there are open Rokhlin towers (B_1, F_1), ..., (B_S, F_S) such that $F_1, F_2, \ldots, F_S$ are $(\mathcal{F}, \varepsilon)$-invariant,

$$B_s\gamma, \quad \gamma \in F_s, s = 1, 2, \ldots, S$$

are mutually disjoint, and

$$\mu(X \setminus \bigsqcup_{s=1}^{S} \bigsqcup_{\gamma\in F_s} B_s\gamma) < \varepsilon, \quad \mu \in \mathcal{M}_1(X, \Gamma).$$

Remark 2.20. If $E \subseteq X$ is a closed set, then $\mu(E) < \varepsilon$ for all $\mu \in \mathcal{M}_1(X, \Gamma)$ if, and only if, the orbit capacity of E is at most ε.

Definition 2.21 ([17])**.** A topological dynamical system (X, Γ) is said to have (λ, m)-Cuntz comparison of open sets, where $\lambda \in (0, +\infty)$ and $m \in \mathbb{N}$, if, for any open sets $E, F \subseteq X$ with

$$\mu(E) \leq \lambda\mu(F), \quad \mu \in \mathcal{M}_1(X, \Gamma),$$

one has

$$\varphi_E \precsim \underbrace{\varphi_F \oplus \cdots \oplus \varphi_F}_{m} \quad \text{in } \mathrm{M}_\infty(\mathrm{C}(X) \rtimes \Gamma).$$

A topological dynamical system is said to have Cuntz comparison of open sets (COS) if it has (λ, m)-Cuntz comparison of open sets for some λ and m.

It follows from Theorem 4.2 and Theorem 5.5 of [16] that

Theorem 2.22. *Any free and minimal dynamical system $(X, \mathbb{Z}^d)$ has the (URP) and (COS).*

3. Some lemmas

In this section, let us develop some lemmas on dimension drop C*-algebras and order zero c.p.c. maps with domain a matrix algebra. Let us start with a simple observation:

Lemma 3.1. *Let a, c be elements of a unital C*-algebra, and assume that $ac = c$ and a is positive. Then,*

$$f(a)c = f(1)c, \quad f \in \mathrm{C}([0, \|a\|]).$$

Proof. If $f(t) = \sum_{k=0}^{n} c_k t^k$, then

$$f(a)c = (\sum_{k=0}^{n} c_k a^k)c = \sum_{k=0}^{n} c_k a^k c = \sum_{k=0}^{n} c_k c = (\sum_{k=0}^{n} c_k)c = f(1)c.$$

The general statement follows from the Weierstrass Theorem. ☐

It is well known that the universal unital C*-algebra generated by v with respect to relations

$$vv^* \perp v^*v \quad \text{and} \quad \|vv^*\| \leq 1$$

is

$$D = \{f : [0, 1] \to \mathrm{M}_2(\mathbb{C}) : f(0) \in \mathbb{C}1_2\},$$

with v corresponding to

$$[0, 1] \ni t \mapsto \begin{pmatrix} 0 & \sqrt{t} \\ 0 & 0 \end{pmatrix}.$$

Using this identification, one has the following lemma:

Lemma 3.2. *Let A be a unital C*-algebra, and let $v \in A$. Consider $a = vv^*$ and $b = v^*v$, and assume that $\|a\| \leq 1$ and $a \perp b$. Define*

$$w = \cos(\frac{\pi}{2}(vv^* + v^*v)) + g(vv^*)v - g(v^*v)v^*,$$

where

$$g(t) = \frac{\sin(\frac{\pi}{2}t)}{\sqrt{t}}, \quad t \in (0, 1].$$

Then $w \in \mathrm{C}^*\{v, 1_A\}$ is a unitary such that

$$b(w^*cw) = (w^*cw)b = w^*cw, \quad \text{if } ac = ca = c, \tag{3.1}$$

and

$$a(wcw^*) = (wcw^*)a = wcw^*, \quad \text{if } bc = cb = c. \tag{3.2}$$

Proof. Noting that $vv^* + v^*v$ is central in $\mathrm{C}^*(1, v)$, $vv^* \perp v^*v$, and
$$v^*h(vv^*) = h(v^*v)v^*, \quad h \in \mathrm{C}_0((0, 1]),$$
one has
$$\begin{aligned} ww^* &= (\cos(\frac{\pi}{2}(vv^* + v^*v)) + g(vv^*)v - g(v^*v)v^*) \\ &\quad (\cos(\frac{\pi}{2}(vv^* + v^*v)) + v^*g(vv^*) - vg(v^*v)) \\ &= \cos^2(\frac{\pi}{2}(vv^* + v^*v)) + g^2(vv^*)vv^* + g^2(v^*v)v^*v \\ &= \cos^2(\frac{\pi}{2}(vv^* + v^*v)) + \sin^2(\frac{\pi}{2}v^*v) + \sin^2(\frac{\pi}{2}vv^*) \\ &= \cos^2(\frac{\pi}{2}(vv^* + v^*v)) + \sin^2(\frac{\pi}{2}(v^*v + vv^*)) \\ &= 1. \end{aligned}$$
The same calculation shows that $w^*w = 1$, and hence w is a unitary.

Let $c \in A$ be an element satisfying
$$ac = ca = c.$$
Note that, for any $f \in \mathrm{C}([0, 1])$, by Lemma 3.1, one has
$$f(a)c = f(1)c = cf(a). \tag{3.3}$$
Therefore,
$$\begin{aligned} w^*cw &= (\cos(\frac{\pi}{2}(vv^* + v^*v)) + v^*g(vv^*) - vg(v^*v))c \\ &\quad (\cos(\frac{\pi}{2}(vv^* + v^*v)) + g(vv^*)v - g(v^*v)v^*) \\ &= (\cos(\frac{\pi}{2}(a)) + v^*g(vv^*))c(\cos(\frac{\pi}{2}(a)) + g(vv^*)v) \\ &= v^*g(vv^*)cg(vv^*)v \\ &= v^*cv, \end{aligned}$$
and hence
$$(w^*cw)b = v^*cvb = v^*cvv^*v = v^*cav = v^*cv = w^*cw$$
and
$$b(w^*cw) = bv^*cv = v^*vv^*cv = v^*acv = v^*cv = w^*cw.$$

A similar calculation shows that
$$a(wcw^*) = (wcw^*)a = wcw^*$$
if $bc = cb = c$. □

The following lemma is crucial in the proof of Proposition 5.4, in which it produces an element that behaves as a lower triangular matrix.

Lemma 3.3. *Let A be a unital C^*-algebra, and let $v_1, v_2, ..., v_n \in A$ be elements satisfying*

- $v_1^*v_1 = v_2^*v_2 = \cdots = v_n^*v_n$,

- *$v_1^*v_1, v_1v_1^*, v_2v_2^*, ..., v_nv_n^*$ are mutually orthogonal, and*
- $\|v_1^*v_1\| = 1$.

Then there is a unitary $w \in A$ satisfies the following properties:

(1) *If $E_1, E_2, ..., E_n \in A$ are mutually orthogonal positive elements such that*
$$[v_1^*v_1, E_i] = 0 \quad \text{and} \quad (v_iv_i^*)E_i = v_iv_i^*, \quad i = 1, 2, ..., n,$$
then
$$wE_i \in \overline{(E_1 + E_i + E_{i+1})A}, \quad 1 \leq i \leq n-1.$$

(2) *If $D \subseteq A$ is a hereditary sub-C*-algebra such that*
$$v_iv_i^* \in D, \quad i = 1, 2, ..., n,$$
and $d \in D$ is an element such that
$$[d, v_1^*v_1] = 0 \quad \text{and} \quad v_n^*d = 0,$$
then
$$wd \in \overline{DA}.$$

(3) *If $c \in A^+$ satisfies*
$$(v_1^*v_1)c = c,$$
then
$$wc \in \overline{(v_1v_1^*)A}.$$

Proof. Consider the universal algebra $\mathcal{A}$ generated by $v_1, v_2, ..., v_n$ with respect to the relations:

- $v_1^*v_1 = v_2^*v_2 = \cdots = v_n^*v_n$,
- $v_1^*v_1, v_1v_1^*, v_2v_2^*, ..., v_nv_n^*$ are mutually orthogonal,
- $\|v_1^*v_1\| = 1$.

It is well known that $\mathcal{A}$ is isomorphic to the dimension drop C*-algebra
$$D_{n+1} := \{f \in \mathrm{C}([0,1], \mathrm{M}_{n+1}) : f(0) = 0_{n+1}\} \cong \mathrm{C}_0((0,1]) \otimes \mathrm{M}_{n+1}(\mathbb{C}) \tag{3.4}$$
with
$$v_i(t) = \sqrt{t} \otimes e_{i,0}, \quad t \in [0,1],\ i = 1, 2, ..., n,$$
where $e_{i,j}$, $i, j = 0, 1, ..., n$, are matrix units of $\mathrm{M}_{n+1}(\mathbb{C})$.

With the identification (3.4), consider the unitary $w \in D_{n+1} + \mathbb{C}1$ defined by
$$w(t) = \begin{cases} 1_{n+1}, & \text{if } t = 0, \\ \begin{pmatrix} 1_{n-k} & & & & & \\ & \cos\frac{\pi}{2}n(t-\frac{k-1}{n}) & 0 & 0 & \cdots & \sin\frac{\pi}{2}n(t-\frac{k-1}{n}) \\ & -\sin\frac{\pi}{2}n(t-\frac{k-1}{n}) & 0 & 0 & \cdots & \cos\frac{\pi}{2}n(t-\frac{k-1}{n}) \\ & & -1 & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \vdots \\ & & & & -1 & 0 \end{pmatrix}, & \text{if } t \in [\frac{k-1}{n}, \frac{k}{n}]. \end{cases}$$

Then the image of w in A, which is still denoted by w, satisfies the properties of the lemma.

Indeed, put

$$w_{i,j} = v_i v_j^*, \quad i,j = 1, ..., n,$$

and

$$w_{0,0} = v_1^* v_1, \quad w_{i,0} = v_i, \quad w_{0,j} = v_j^*, \quad i,j = 1,2,...,n.$$

Then

$$w = 1 + \sum_{i=0}^{n} g_{i,i}(w_{i,i}) + \sum_{i=1}^{n} g_{i,i-1}(w_{i,i}) w_{i,i-1} + \sum_{i=0}^{n-1} g_{i,n}(w_{i,i}) w_{i,n},$$

for some functions $g_{i,j} \in \mathrm{C}_0((0,1])$. Note that

$$g_{0,0}(1) = -1. \tag{3.5}$$

Let $E_1, E_2, ..., E_n$ be mutually orthogonal positive elements of A satisfying

$$w_{i,i} E_i = w_{i,i}, \quad i = 1,2,...,n, \tag{3.6}$$

and

$$[w_{0,0}, E_i] = 0. \tag{3.7}$$

Note that, by (3.6) and polar-decomposition, one has

$$v_j^* E_i = v_j'(v_j v_j^*)^{\frac{1}{2}} E_i = v_j' w_{j,j}^{\frac{1}{2}} E_i = v_j' w_{j,j}^{\frac{1}{2}} E_j E_i = 0, \quad j \neq i, \ 1 \leq i,j \leq n,$$

where v_j' is a partial isometry in the enveloping von Neumann algebra. Then, for any $1 \leq i \leq n-1$,

$$w_{i_1,i_2} E_i = v_{i_1} v_{i_2}^* E_i = 0, \quad i_2 \neq i, \ 1 \leq i_1, i_2 \leq n$$

and hence

$$\begin{aligned} wE_i &= (1 + \sum_{j=0}^{n} g_{j,j}(w_{j,j}) + \sum_{j=1}^{n} g_{j,j-1}(w_{j,j}) w_{j,j-1} + \sum_{j=0}^{n-1} g_{j,n}(w_{j,j}) w_{j,n}) E_i \\ &= E_i + g_{0,0}(w_{0,0}) E_i + g_{i,i}(w_{i,i}) E_i + g_{1,0}(w_{1,1}) w_{1,0} E_i + g_{i+1,i}(w_{i+1,i+1}) w_{i+1,i} E_i. \end{aligned}$$

By (3.7),

$$g_{0,0}(w_{0,0}) E_i = E_i g_{0,0}(w_{0,0}) \in E_i A.$$

By (3.6),

$$g_{i,i}(w_{i,i}) E_i = E_i g_{i,i}(w_{i,i}) \in E_i A,$$

$$g_{1,0}(w_{1,1}) w_{1,0} E_i = E_1 g_{1,0}(w_{1,1}) w_{1,0} E_i \in E_1 A,$$

and

$$g_{i+1,i}(w_{i+1,i+1}) w_{i+1,i} E_i = E_{i+1} g_{i+1,i}(w_{i+1,i+1}) w_{i+1,i} E_i \in E_{i+1} A.$$

Therefore,

$$wE \in \overline{(E_1 + E_i + E_{i+1})A},$$

and this proves Property (1).

For Property (2), let D be a hereditary sub-C*-algebra such that

$$w_{i,i} \in D, \quad i = 1,2,...,n,$$

and let $d \in D$ be an element satisfying

$$[d, w_{0,0}] = 0 \quad \text{and} \quad w_{0,n}d = 0.$$

Then

$$\begin{aligned} wd &= (1 + \sum_{i=0}^{n} g_{i,i}(w_{i,i}) + \sum_{i=1}^{n} g_{i,i-1}(w_{i,i})w_{i,i-1} + \sum_{i=0}^{n-1} g_{i,n}(w_{i,i})w_{i,n})d \\ &= d + g_{0,0}(w_{0,0})d + \sum_{i=1}^{n} g_{i,i}(w_{i,i})d + \sum_{i=1}^{n} g_{i,i-1}(w_{i,i})w_{i,i-1}d + \sum_{i=1}^{n-1} g_{i,n}(w_{i,i})w_{i,n}d \\ &\in \overline{DA}. \end{aligned}$$

For Property (3), let $c \in A$ be a positive element such that

$$w_{0,0}c = c.$$

Then, by Lemma 3.1 and (3.5), one has

$$\begin{aligned} wc &= (1 + \sum_{i=0}^{n} g_{i,i}(w_{i,i}) + \sum_{i=1}^{n} g_{i,i-1}(w_{i,i})w_{i,i-1} + \sum_{i=0}^{n-1} g_{i,n}(w_{i,i})w_{i,n})c \\ &= c + g_{0,0}(w_{0,0})c + g_{1,0}(w_{1,1})w_{1,0}c \\ &= (1 + g_{0,0}(1))c + g_{1,0}(w_{1,1})w_{1,0}c \\ &= g_{1,0}(w_{1,1})w_{1,0}c \in \overline{w_{1,1}A}. \end{aligned}$$

This proves the lemma. □

Lemma 3.4. *Let A be a unital C*-algebra, and let $\phi : \mathrm{M}_n(\mathbb{C}) \to A$ be an extendable order zero c.p.c. map. Denote by $e_i = \phi(e_{i,i})$ and $h = \phi(1_n)$. Then, for any permutation $\sigma : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$, there is a unitary $u \in A$ such that*

$$[u, h] = 0 \quad \text{and} \quad u^* e_i u = e_{\sigma(i)}, \quad i = 1, 2, ..., n.$$

Proof. Since ϕ is extendable, there is an order zero map $\tilde{\phi} : \mathrm{M}_n(\mathbb{C}) \to A$ and $\delta > 0$ such that $\phi = f_\delta(\tilde{\phi})$. Note that

$$C := \mathrm{C}^*\{\tilde{\phi}(\mathrm{M}_n(\mathbb{C}))\} \cong \{f : [0, 1] \to \mathrm{M}_n(\mathbb{C}) : f(0) = 0_n\} \cong \mathrm{C}_0((0, 1]) \otimes \mathrm{M}_n(\mathbb{C})$$

with $e_i = f_\delta \otimes e_{i,i}$, $i = 1, 2, ..., n$, under the isomorphism. Denote by $U \in \mathrm{M}_n(\mathbb{C})$ the permutation unitary matrix such that

$$U^* e_{i,i} U = e_{\sigma(i),\sigma(i)}, \quad i = 1, 2, ..., n.$$

Since the unitary group of $\mathrm{M}_n(\mathbb{C})$ is path connected, there is a continuous path of unitaries

$$[0, \frac{\delta}{2}] \ni t \mapsto U_t \in \mathrm{M}_n(\mathbb{C})$$

such that $U_0 = 1_n$ and $U_{\delta/2} = U$. Set

$$u : t \mapsto u(t) = \begin{cases} U_t, & t \in [0, \frac{\delta}{2}], \\ U, & t \in [\frac{\delta}{2}, 1]. \end{cases}$$

Then the unitary $u \in C + \mathbb{C}1_A \subseteq A$ has the desired property. □

4. Nilpotent elements, order zero maps, and limits of invertible elements

An element a of a C*-algebra is said to be nilpotent if $a^n = 0$ for some $n \in \mathbb{N}$. It is well known that if a is nilpotent, then $a + \varepsilon 1_A$ is invertible for any non-zero ε; in fact,

$$(a+\varepsilon 1_A)^{-1} = \frac{1_A}{\varepsilon} - \frac{a}{\varepsilon^2} + \frac{a^2}{\varepsilon^3} + \cdots + (-1)^{n-1}\frac{a^{n-1}}{\varepsilon^n} + \cdots, \tag{4.1}$$

where the infinite series is eventually zero, as a is nilpotent. Hence any nilpotent element is in the closure of invertible elements.

The following lemma is a modified version of Lemma 4.2 of [2].

Lemma 4.1 (c.f. Lemma 4.2 of [2])**.** *Let A be a finite unital C^*-algebra and let $a \in A$. Suppose there exist positive elements b_1, b_2, c_1, c_2 such that*

(1) $b_1 + b_2 = 1_A$,
(2) $c_1 + c_2 = 1_A$,
(3) $\mathrm{C}^*\{b_1, b_2, c_1, c_2\}$ *is commutative,*
(4) $b_1 c_1 = c_1$,
(5) $b_2 c_2 = b_2$,
(6) $c_1 b_2 = 0$,
(7) *there are unitaries $u, v \in A$ such that*
$$u(b_2 a(b_1 - c_1) + b_2 a b_2)v \quad \text{and} \quad u(b_2 a b_2)v$$
are nilpotent,
$$v(u(b_2 a b_2)v)^n u \in \overline{b_2 A b_2}, \quad n = 1, 2, ..., \tag{4.2}$$
and
$$c_1 v u b_2 = 0$$
(8) $b_1 a = 0$.

Then $a \in \overline{\mathrm{GL}(A)}$

Proof. The proof is the similar to that of Lemma 5.2 of [2], but with $c_3 = b_3 = 0$.

Fix an arbitrary $\varepsilon > 0$ for the time being, and note that
$$1_A = c_1 + (b_1 - c_1) + b_2.$$
Put
$$a_{3,1} = b_2 a c_1$$
$$a_{3,2} = b_2 a (b_1 - c_1)$$
$$a_{3,3} = b_2 a b_2.$$
Then, by (1) and (8),
$$a_{3,1} + a_{3,2} + a_{3,3} = b_2 a(c_1 + (b_1 - c_1) + b_2) = b_2 a(b_1 + b_2) = b_2 a = (b_1 + b_2)a = a.$$

Note that
$$a_{3,3} = u^{-1} a'_{3,3} v^{-1},$$
where
$$a'_{3,3} := u a_{3,3} v = u(b_2 a b_2)v$$

is nilpotent (by 7).

Put

$$t_0 = u^{-1}(a'_{3,3} + \varepsilon 1_A)v^{-1}.$$

Then t_0 is invertible and

$$\|t_0 - a_{3,3}\| = \left\|\varepsilon u^{-1}v^{-1}\right\| = \varepsilon. \tag{4.3}$$

Using (4.1), write

$$t_0^{-1} = v(\frac{1_A}{\varepsilon} - \frac{a'_{3,3}}{\varepsilon^2} + \frac{(a'_{3,3})^2}{\varepsilon^3} + \cdots + (-1)^{n-1}\frac{(a'_{3,3})^{n-1}}{\varepsilon^n} + \cdots)u = t_2 + \frac{vu}{\varepsilon},$$

where, by (4.2),

$$t_2 := -\frac{va'_{3,3}u}{\varepsilon^2} + \frac{v(a'_{3,3})^2u}{\varepsilon^3} + \cdots + (-1)^{n-1}\frac{v(a'_{3,3})^{n-1}u}{\varepsilon^n} + \cdots \in \overline{b_2Ab_2}.$$

(Note that the series above are eventually zero.) Therefore, together with

$$c_1b_2 = 0 \quad \text{and} \quad c_1vub_2 = 0,$$

one has

$$a_{3,1}t_0^{-1}a_{3,1} = b_2ac_1(t_2 + \frac{vu}{\varepsilon})b_2ac_1 = 0 \tag{4.4}$$

and

$$a_{3,1}t_0^{-1}a_{3,2} = b_2ac_1(t_2 + \frac{vu}{\varepsilon})b_2a(b_1 - c_1) = 0. \tag{4.5}$$

Therefore, by (4.4), one has

$$(a_{3,1} + t_0)t_0^{-1}(1_A - a_{3,1}t_0^{-1}) = a_{3,1}t_0^{-1} - a_{3,1}t_0^{-1}a_{3,1}t_0^{-1} + 1_A - a_{3,1}t_0^{-1} = 1_A \tag{4.6}$$

and

$$t_0^{-1}(1_A - a_{3,1}t_0^{-1})(a_{3,1} + t_0) = t_0^{-1}a_{3,1} + 1_A - t_0^{-1}a_{3,1}t_0^{-1}a_{3,1} - t_0^{-1}a_{3,1} = 1_A.$$

In particular, the element $(a_{3,1} + t_0)$ is invertible with

$$(a_{3,1} + t_0)^{-1} = t_0^{-1}(1_A - a_{3,1}t_0^{-1}).$$

Then, together with (4.5), one has

$$\begin{aligned} t_0^{-1}(1_A - a_{3,1}t_0^{-1})(a_{3,1} + a_{3,2} + t_0) &= (a_{3,1} + t_0)^{-1}(a_{3,1} + a_{3,2} + t_0) \\ &= 1_A + (a_{3,1} + t_0)^{-1}a_{3,2} \\ &= 1_A + t_0^{-1}(1 - a_{3,1}t_0^{-1})a_{3,2} \\ &= 1_A + t_0^{-1}a_{3,2}. \end{aligned} \tag{4.7}$$

Consider

$$\begin{aligned} t_1 &:= 1_A + t_0^{-1}a_{3,2} \\ &= t_0^{-1}(t_0 + a_{3,2}) \\ &= t_0^{-1}(u^{-1}(a'_{3,3} + \varepsilon 1_A)v^{-1} + u^{-1}a'_{3,2}v^{-1}) \\ &= t_0^{-1}u^{-1}(a'_{3,3} + a'_{3,2} + \varepsilon 1_A)v^{-1}, \end{aligned} \tag{4.8}$$

where

$$a'_{3,2} = ua_{3,2}v = u(b_2a(b_1 - c_1))v.$$

Since

$$a'_{3,3} + a'_{3,2} = u((b_2ab_2) + b_2a(b_1 - c_1))v$$

is nilpotent, one has that t_1 is invertible.

Put

$$y = (a_{3,1} + t_0)t_1.$$

Since $a_{3,1} + t_0$ and t_1 are invertible, one has

$$y \in \mathrm{GL}(A).$$

Applying (4.6) in the third step, (4.7) in the fifth step, definition of t_1 (see (4.8)) in the sixth step, and (4.3) in the last step, one has

$$\begin{aligned} \|a - y\| &= \|a_{3,1} + a_{3,2} + a_{3,3} - (a_{3,1} + t_0)t_1\| \\ &\leq \|a_{3,3} - t_0\| + \|1_A(a_{3,1} + a_{3,2} + t_0) - (a_{3,1} + t_0)t_1\| \\ &\leq \|a_{3,3} - t_0\| + \left\|(a_{3,1} + t_0)t_0^{-1}(1_A - a_{3,1}t_0^{-1})(a_{3,1} + a_{3,2} + t_0) - (a_{3,1} + t_0)t_1\right\| \\ &\leq \|a_{3,3} - t_0\| + \|a_{3,1} + t_0\| \left\|t_0^{-1}(1_A - a_{3,1}t_0^{-1})(a_{3,1} + a_{3,2} + t_0) - t_1\right\| \\ &= \|a_{3,3} - t_0\| + \|a_{3,1} + t_0\| \left\|1_A + t_0^{-1}a_{3,2} - t_1\right\| \\ &= \|a_{3,3} - t_0\| = \varepsilon. \end{aligned}$$

Since ε is arbitrary, one has $a \in \overline{\mathrm{GL}(A)}$, as desired. □

Lemma 4.2. *Let A be a unital C^*-algebra, and let $\phi : \mathrm{M}_n(\mathbb{C}) \to A$ be an extendable order zero c.p.c map. Set*

$$h = \phi(1_n) \quad \text{and} \quad e_i = \phi(e_{i,i}), \quad i = 1, 2, ..., n,$$

and set

$$b_2 = f_\delta(h), \quad b_1 = 1 - b_2, \quad c_2 = f_{\delta/2}(h), \quad c_1 = 1 - f_{\delta/2}(h),$$

for some $\delta \in (0, 1)$.

Then, for any $m \in \mathbb{N}$ with $m \leq n$ and for some

$$1 \leq i_1 < i_2 < \cdots < i_m \leq n,$$

there are unitaries $u, v \in A$ satisfying the following properties:

1. *$c_1vub_2 = 0$, and*
2. *if $a \in \overline{c_2Ac_2}$ satisfies*

$$e_iae_j = 0, \quad \text{if } j - i > d,$$

for some $d \in \mathbb{N}$ with $d < m$, and

$$e_{i_j}a = ae_{i_j} = 0, \quad 1 \leq j \leq m.$$

Then uav is nilpotent. Moreover, if $a \in \overline{b_2Ab_2}$, then

$$v(uav)^ku \in \overline{b_2Ab_2}, \quad k = 1, 2, ...$$

Proof. With the given m and $i_1, i_2, ..., i_m$, define the permutation

$$\sigma : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$$

by stretching $\{1, 2, ..., n-m\}$ to $\{1, 2, ..., n\} \setminus \{i_1, i_2, ..., i_m\}$, and then moves $\{n-m+1, ..., n\}$ to fill $\{i_1, i_2, ..., i_m\}$. More precisely, write

$$\begin{cases} I_0 = \{1 \leq i \leq n-m : i < i_1\}, \\ I_1 = \{1 \leq i \leq n-m : i_1 \leq i,\ i+1 < i_2\}, \\ \cdots \\ I_{m-1} = \{1 \leq i \leq n-m : i_{m-1} \leq i+m-2,\ i+m-1 < i_m\}, \\ I_m = \{1 \leq i \leq n-m : i_m \leq i+m-1\}, \end{cases}$$

and note that $\{1, 2, ..., n-m\} = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_m$ (some of I_k, $k = 1, ..., m$, might be empty). Then

$$\sigma(i) = \begin{cases} i+k, & \text{if } i \in I_k, \\ i_{i-n+m}, & n-m+1 \leq i \leq n. \end{cases}$$

Note that for any $d \in \mathbb{N}$,

$$\sigma(j) - \sigma(i) > d, \quad \text{if } 1 \leq i, j \leq n-m \text{ and } j - i > d. \tag{4.9}$$

Since ϕ is extendable, by Lemma 3.4, there is a unitary $w_1 \in A$ such that

$$[w_1, h] = 0 \quad \text{and} \quad w_1^* e_i w_1 = e_{\sigma(i)}, \quad i = 1, 2, ..., n.$$

By Lemma 3.4 again, there is a unitary $w_2 \in A$ satisfying

$$[w_2, h] = 0$$

and

$$w_2^* e_i w_2 = \begin{cases} e_{i+n-m}, & 1 \leq i \leq m, \\ e_{i-m}, & m+1 \leq i \leq n. \end{cases}$$

Then, the unitaries

$$u := w_2 w_1 \quad \text{and} \quad v := w_1^*$$

satisfy the property of the lemma.

Indeed, since w_1 and w_2 commute with h and $c_1 b_2 = 0$, one has that

$$c_1(vu)b_2 = c_1(w_1^* w_2 w_1)b_2 = (w_1^* w_2 w_1)(c_1 b_2) = 0.$$

Let $a \in \overline{c_2 A c_2}$ satisfy

$$e_i a e_j = 0, \quad \text{if } j - i > d$$

for some natural number $d < m$, and

$$e_{i_j} a = a e_{i_j} = 0, \quad 1 \leq j \leq m.$$

Consider the element $w_1 a w_1^*$. Note that, for any $n-m+1 \leq i \leq n$,

$$\sigma(i) \in \{i_1, i_2, ..., i_m\}.$$

Hence

$$e_i(w_1 a w_1^*) = w_1(e_{\sigma(i)} a) w_1^* = 0, \quad n-m+1 \leq i \leq n \tag{4.10}$$

and

$$(w_1aw_1^*)e_i = w_1(ae_{\sigma(i)})w_1^* = 0, \quad n-m+1 \leq i \leq n. \tag{4.11}$$

Also note that, for any $1 \leq i, j \leq n-m$ satisfying $j-i > d$, by (4.9), one has that $\sigma(j) - \sigma(i) > d$, and hence

$$e_i(w_1aw_1^*)e_j = w_1(e_{\sigma(i)}ae_{\sigma(j)})w_1^* = 0.$$

Together with (4.10) and (4.11), one has

$$e_i(w_1aw_1^*)e_j = 0, \quad j-i > d,\ 1 \leq i, j \leq n. \tag{4.12}$$

Consider the element $uav = w_2w_1aw_1^*$, and note that for any $1 \leq i, j \leq n$ with $j \geq i$,

- if $1 \leq i \leq m$, then $n-m+1 \leq i+n-m \leq n$, and by (4.10),

$$e_i(w_2w_1aw_1^*)e_j = w_2(e_{i+n-m}w_1aw_1^*)e_j = 0;$$

- if $m+1 \leq i \leq n$, then $j-(i-m) \geq m > d$, and hence, by (4.12),

$$e_i(w_2w_1aw_1^*)e_j = w_2(e_{i-m}w_1aw_1^*e_j) = 0.$$

That is,

$$e_i(uav)e_j = 0, \quad j \geq i. \tag{4.13}$$

Consider

$$\tilde{e}_i := f_{\frac{\delta}{4}}(e_i), \quad i = 1, 2, ..., n,$$

and it follows from (4.13) that

$$\tilde{e}_i(uav)\tilde{e}_j = 0, \quad j \geq i.$$

Since $uav \in \overline{c_2Ac_2}$, one has

$$uav = (\tilde{e}_1 + \cdots + \tilde{e}_n)uav(\tilde{e}_1 + \cdots + \tilde{e}_n) = \sum_{i,j=1}^{n} \tilde{e}_i(uav)\tilde{e}_j = \sum_{i>j} \tilde{e}_i(uav)\tilde{e}_j.$$

That is, there is a decomposition

$$uav = \sum_{i>j} a_{i,j}^{(1)},$$

where $a_{i,j}^{(1)} \in \overline{\tilde{e}_iA\tilde{e}_j}$. A direct calculation shows that

$$(uav)^2 = \sum_{i>k>j} a_{i,k}a_{k,j} = \sum_{i>k-1} a_{i,j}^{(2)},$$

where $a_{i,j}^{(2)} \in \overline{\tilde{e}_iA\tilde{e}_j}$.

Repeating n times, one has $(uav)^n = 0$. Hence uav is nilpotent.

If, moreover, $a \in \overline{b_2Ab_2}$, since u and v commute with h (and hence commute with b_2), one has

$$v(uav)^ku \in v(u(\overline{b_2Ab_2})v)^ku \subseteq \overline{b_2Ab_2}, \quad k = 1, 2, ...,$$

as desired. $\square$

Proposition 4.3. *Let A be a unital C*-algebra, and let $a \in A$. If there exist an extendable order zero c.p.c. map $\phi : \mathrm{M}_n(\mathbb{C}) \to A$, natural numbers $d < m$ and $1 \leq i_1 < i_2 < \cdots < i_m \leq n$, such that*

1. *$(1-h)a = 0$, where $h = \phi(1_n)$,*
2. *$e_{i_j}a = ae_{i_j} = 0$, $j = 1, 2, ..., m$, where $e_i = \phi(e_{i,i})$, and*
3. *$e_i a e_j = 0$, if $j - i > d$.*

Then $a \in \overline{\mathrm{GL}(A)}$.

Proof. Pick an arbitrary $\delta \in (0,1)$, and consider

$$b_2 = f_\delta(h), \quad c_2 = f_{\delta/2}(h), \quad b_1 = 1 - f_\delta(h), \quad \text{and} \quad c_1 = 1 - f_{\delta/2}(h).$$

Note that

$$c_2 b_2 = b_2, \quad b_1 c_1 = c_1, \quad \text{and} \quad c_1 b_2 = 0.$$

Since $(1-h)a = 0$, one has $a = ha$, and

$$b_2 a = f_\delta(h)a = f_\delta(1)a = a.$$

Hence $b_1 a = 0$.

Consider the elements

$$b_2 a b_2 \quad \text{and} \quad b_2 a(b_1 - c_1) + b_2 a b_2,$$

and note that both of them are in $\overline{c_2 A c_2}$. Also note that (since h commutes with e_i, $i = 1, 2, ..., n$)

$$e_i b_2 a b_2 e_j = b_2 e_i a e_j b_2 = 0, \quad j - i > d$$

and

$$e_{i_j}(b_2 a b_2) = b_2 e_{i_j} a b_2 = 0 = b_2 a e_{i_j} b_2 = (b_2 a b_2) e_{i_j}, \quad j = 1, 2, ..., m;$$

and the same argument shows that

$$e_i(b_2 a(b_1 - c_1) + b_2 a b_2)e_j = 0, \quad j - i > d,$$

and

$$e_{i_j}(b_2 a(b_1 - c_1) + b_2 a b_2) = 0 = (b_2 a(b_1 - c_1) + b_2 a b_2)e_{i_j}, \quad j = 1, 2, ..., m.$$

Let $u, v \in A$ be the unitaries obtained by applying Lemma 4.2 to ϕ, δ, m and $i_1, i_2, ..., i_m$. Then, by Lemma 4.2, one has

$$c_1 v u b_2 = 0,$$

$$u(b_2 a b_2)v \quad \text{and} \quad u(b_2 a(b_1 - c_1) + b_2 a b_2)v$$

are nilpotent, and

$$v(u b_2 a b_2 v)^n u \in \overline{b_2 A b_2}, \quad n = 1, 2, ...$$

Then, by Lemma 4.1, one has $a \in \overline{\mathrm{GL}(A)}$, as desired. □

5. Property (D) and stable rank one

Definition 5.1. Let A be a unital C*-algebra. An element $a \in A$ is said to be a $\mathcal{D}_0$-operator if there exists a nonzero positive element $b \in A$ satisfying

$$ba = ab = 0,$$

and there exists an order zero c.p.c. map

$$\phi : \mathrm{M}_{pq}(\mathbb{C}) \to A,$$

where $p, q \in \mathbb{N}$, and there exist $r, l \in \mathbb{N}$ such that, with

$$e_i := \phi(e_{i,i}), \quad i = 1, 2, ..., pq,$$

$$s_k := e_{(k-1)p+1} + \cdots + e_{(k-1)p+r}, \quad k = 1, ..., q,$$

and

$$E_k := e_{(k-1)p+1} + \cdots + e_{(k-1)p+p}, \quad k = 1, ..., q,$$

one has

(1) $E_{k_1} a E_{k_2} = 0$, $k_2 - k_1 \geq l$, $1 \leq k_1, k_2 \leq q$;
(2) $qr > (l+1)p$;
(3) for each $k = 1, 2, ..., q$, there are positive elements c_k, d_k with norm 1 such that
  (a) $c_k, d_k \in \overline{bAb}$,
  (b) $c_k \perp s_k$ and $c_k \perp d_k$,
  (c) $c_k E_k = c_k$ and $d_k E_k = d_k$, and
  (d) $s_k \precsim c_k$ and $1_A - h \precsim d_k$, where $h := \phi(1_{pq})$.

Recall that

Definition 5.2. Let A be a C*-algebra, then, define

$$\mathrm{ZD}(A) := \{a \in A : d_1 a = a d_2 = 0 \text{ for some } d_1, d_2 \in A^+ \setminus \{0\}\}.$$

And define

Definition 5.3. The C*-algebra A is said to have Property (D) if for any $a \in \mathrm{ZD}(A)$ and any $\varepsilon > 0$, there are unitaries $u_1, u_2 \in A$ and a $\mathcal{D}_0$-operator $a' \in A$ such that $\|u_1 a u_2 - a'\| < \varepsilon$.

It turns out that any $\mathcal{D}_0$-operator is in the norm closure of invertible elements.

Proposition 5.4. *Let $a \in A$ be a $\mathcal{D}_0$-operator of a unital C*-algebra A. Then $a \in \overline{\mathrm{GL}(A)}$.*

Proof. Let a be a $\mathcal{D}_0$-operator. Then there is $b \in A^+ \setminus \{0\}$ such that

$$ba = ab = 0; \tag{5.1}$$

and there exists an order zero c.p.c. map

$$\phi' : \mathrm{M}_{pq}(\mathbb{C}) \to A,$$

where $p, q \in \mathbb{N}$, and there exist $r, l \in \mathbb{N}$ such that, with

$$e_i' := \phi'(e_{i,i}), \quad i = 1, 2, ..., pq,$$

$$s_k := e'_{(k-1)p+1} + \cdots + e'_{(k-1)p+r}, \quad k = 1, ..., q,$$

and

$$E_k := e'_{(k-1)p+1} + \cdots + e'_{(k-1)p+p}, \quad k = 1, ..., q,$$

one has

(1) $E_{k_1} a E_{k_2} = 0$, $k_2 - k_1 \geq l$, $1 \leq k_1, k_2 \leq q$;
(2) $qr > (l+1)p$;
(3) there are positive elements c_k, d_k, $k = 1, 2, ..., q$, with norm 1 such that
  (a) $c_k, d_k \in \overline{bAb}$,
  (b) $c_k \perp s_k$ and $c_k \perp d_k$,
  (c) $c_k E_k = c_k$ and $d_k E_k = d_k$, and
  (d) $s_k \precsim c_k$ and $1_A - h' \precsim d_k$, where $h' := \phi'(1_{pq})$.

With above, one asserts that there exist an extendable order zero c.p.c. map

$$\phi : \mathrm{M}_{pq}(\mathbb{C}) \to A$$

and unitaries $u, w \in A$ such that, with

$$h := \phi(1_{pq}) \quad \text{and} \quad e_i := \phi(e_{i,i}), \quad i = 1, 2, ..., pq,$$

then

- $(1_A - h)(uw^*au^*) = 0$,
- $e_i(uw^*au^*)e_j = 0$, $j - i > (l+1)q$, and
- there are $e_{i_1}, ..., e_{i_{qr}}$ such that

$$e_{i_j}(uw^*au^*) = (uw^*au^*)e_{i_j} = 0, \quad j = 1, 2, ..., qr.$$

It then follows from Proposition 4.3 (with $d = (l+1)p$ and $m = qr$) and Condition 2 that

$$uw^*au^* \in \overline{\mathrm{GL}(A)}.$$

Since u, w are unitaries, one has that $a \in \overline{\mathrm{GL}(A)}$, and the proposition follows.

Let us show the assertion. By Condition 3a and (5.1)

$$c_k a = a c_k = d_k a = a d_k = 0, \quad k = 1, ..., q. \tag{5.2}$$

Consider the positive element $f_{\frac{1}{2}}(h')$, and note that $1_A - f_{\frac{1}{2}}(h') \precsim 1_A - h'$. Then, by Condition 3d, one has

$$1_A - f_{\frac{1}{2}}(h') \precsim d_k, \quad k = 1, 2, ..., q,$$

and therefore, by Proposition 2.4(iv) of [23] (see Lemma 2.10), there are

$$v_1, v_2, ..., v_q \in A$$

such that

$$v_k^* v_k = f_{\frac{1}{2}}(1_A - f_{\frac{1}{2}}(h')) \quad \text{and} \quad v_k v_k^* \in \mathrm{Her}(d_k) \subseteq \mathrm{Her}(b), \quad k = 1, ..., q. \tag{5.3}$$

It follows from Condition 3c that $f_{\frac{1}{2}}(1_A - f_{\frac{1}{2}}(h')) \perp d_k$; then, using Condition 3c again, one has

$$f_{\frac{1}{2}}(1_A - f_{\frac{1}{2}}(h')) \perp v_k v_k^* \quad \text{and} \quad v_k v_k^* \in \mathrm{Her}(E_k), \quad 1 \leq k \leq q,$$

and hence

$$v_1^*v_1,\ v_1v_1^*,\ v_2v_2^*, ..., v_qv_q^*$$

are mutually orthogonal. Applying Lemma 3.3 to $v_1, v_2, ..., v_q$, one obtains the unitary $w \in A$ which satisfies the properties of Lemma 3.3.

By Condition 3d,

$$s_k \precsim c_k, \quad k = 1, ..., q. \tag{5.4}$$

Since $s_k, c_k \in \overline{E_kAE_k}$, one may assume that the Cuntz sub-equivalences (5.4) hold in the hereditary sub-C*-algebra $\overline{E_kAE_k}$. By Proposition 2.4(iv) of [23] (see Lemma 2.10), there is $z_k \in \overline{E_kAE_k}$ such that

$$z_k^*z_k = f_{\frac{1}{8}}(s_k) \quad \text{and} \quad z_kz_k^* \in \mathrm{Her}(c_k) \subseteq \mathrm{Her}(b). \tag{5.5}$$

Note that, by Condition (3c),

$$z_k \in \overline{f_{\frac{1}{8}}(E_k)Af_{\frac{1}{8}}(E_k)}. \tag{5.6}$$

By Condition 3b,

$$z_k^*z_k = f_{\frac{1}{8}}(s_k) \perp \overline{c_kAc_k} \ni z_kz_k^*.$$

Since $f_{\frac{1}{8}}(s_k)f_{\frac{1}{4}}(s_k) = f_{\frac{1}{4}}(s_k)$, applying Lemma 3.2 to $v = z_k$, one has that, with

$$u_k := \cos(\frac{\pi}{2}(z_kz_k^* + z_k^*z_k)) + z_k^*g(z_kz_k^*) - z_kg(z_k^*z_k), \tag{5.7}$$

where $g(t) = \sin(\pi t/2)/\sqrt{t}$, $t \in (0, 1]$, then $u_k \in \mathrm{C}^*(z_k, 1)$ is a unitary such that

$$u_k^*f_{\frac{1}{4}}(s_k)u_k \in \mathrm{Her}(c_k) \subseteq \mathrm{Her}(b).$$

By (5.6), one has that $u_k \in \overline{f_{\frac{1}{8}}(E_k)Af_{\frac{1}{8}}(E_k)} + \mathbb{C}1_A$ and hence

$$u_k^*E_ku_k \in \overline{E_kAE_k},$$

$$u_k^*au_k = a, \quad a \in \overline{E_{k'}AE_{k'}},\ k \neq k'.$$

$$u_k^*f_{\frac{1}{8}}(E_k)u_k \in \overline{f_{\frac{1}{8}}(E_k)Af_{\frac{1}{8}}(E_k)} \quad \text{and} \quad [u_k, f_{\frac{1}{16}}(E_k)] = 0. \tag{5.8}$$

In particular, with

$$u := \prod_{k=1}^{q} u_k,$$

one has

$$u^*E_ku \in \overline{E_kAE_k}, \quad 1 \leq k \leq q, \tag{5.9}$$

$$[u, f_{\frac{1}{16}}(E_k)] = 0, \quad 1 \leq k \leq q, \tag{5.10}$$

and

$$u^*f_{\frac{1}{4}}(s_k)u \in \mathrm{Her}(c_k) \subseteq \mathrm{Her}(b), \quad 1 \leq k \leq q. \tag{5.11}$$

Consider the positive element $1_A - f_{\frac{1}{4}}(h')$, and note that

$$(1_A - f_{\frac{1}{4}}(h'))v_1^*v_1 = (1_A - f_{\frac{1}{4}}(h'))f_{\frac{1}{2}}(1_A - f_{\frac{1}{2}}(h')) = 1_A - f_{\frac{1}{4}}(h').$$

It follows from Lemma 3.3(3) that

$$(1_A - f_{\frac{1}{4}}(h'))w^* \in \overline{A(v_1v_1^*)} \subseteq \overline{Ad_1}$$

and therefore, by (5.2),

$$(1_A - f_{\frac{1}{4}}(h'))w^*a = 0. \tag{5.12}$$

Since $v_1^*v_1, E_k \in \mathrm{C}^*(e'_1, ..., e'_{pq})$, which is commutative, together with Condition 3c and (5.3), one has

$$[v_1^*v_1, E_k] = 0 \quad \text{and} \quad (v_kv_k^*)E_k = v_kv_k^*, \quad k = 1, 2, ..., q.$$

It then follows from Lemma 3.3(1) that

$$E_kw^* \in \overline{A(E_1 + E_k + E_{k+1})}, \quad k = 1, 2, ..., q-1,$$

and hence, by Condition 1, for any $k_2 - k_1 \geq l + 1$ where $1 \leq k_1, k_2 \leq q$, one has that $E_{k_1}w^*aE_{k_2} = 0$, and then, by (5.9),

$$E_{k_1}(uw^*au^*)E_{k_2} = u(u^*E_{k_1}u)w^*a(u^*E_{k_2}u)u^* \in u\overline{\overline{E_{k_1}AE_{k_1}}w^*a\overline{E_{k_2}AE_{k_2}}}u^* = \{0\}.$$

In particular,

$$f_{\frac{1}{4}}(E_{k_1})(uw^*au^*)f_{\frac{1}{4}}(E_{k_2}) = 0, \quad k_2 - k_1 \geq l + 1. \tag{5.13}$$

Consider

$$c := \sum_{k=1}^{q} c_k.$$

Note that $c \in \overline{bAb}$. By Condition 3c, one has that $ch' = c$; in particular, $[c, h'] = 0$ and hence, by (5.3), $[c, v_1^*v_1] = 0$. Also note $c \perp v_qv_q^* \in \mathrm{Her}(d_q)$. Then, it follows from Lemma 3.3(2) (with c in the place of d and $\overline{bAb}$ in the place of D) that

$$cw^* \in \overline{A(\overline{bAb})}. \tag{5.14}$$

By (5.11), for each $k = 1, 2, ..., q$ and $i = 1, ..., r$, one has

$$u^*f_{\frac{1}{4}}(e'_{(k-1)p+i})u \leq u^*f_{\frac{1}{4}}(s_k)u \in \mathrm{Her}(c_k) \subseteq \overline{cAc} \subseteq \overline{bAb}. \tag{5.15}$$

Then, together with (5.14),

$$(u^*f_{\frac{1}{4}}(e'_{(k-1)p+i})u)w^* \in (\overline{cAc})w^* \subseteq \overline{cAcw^*} \subseteq \overline{A(\overline{bAb})}.$$

Hence

$$f_{\frac{1}{4}}(e'_{n(k-1)+i})(uw^*au^*) = u((u^*f_{\frac{1}{4}}(e'_{(k-1)p+i})u)w^*)au^* \in u\overline{A(\overline{bAb})}au^* = \{0\};$$

and, on the other hand, by (5.15),

$$(uw^*au^*)f_{\frac{1}{4}}(e'_{(k-1)p+i}) = uw^*(a(u^*f_{\frac{1}{4}}(e'_{(k-1)p+i})u))u^* \in uw^*(a(\overline{bAb}))u^* = \{0\}.$$

That is, for any $k = 1, ..., q$ and $i = 1, ..., r$,

$$f_{\frac{1}{4}}(e'_{(k-1)p+i})(uw^*au^*) = 0 \quad \text{and} \quad (uw^*au^*)f_{\frac{1}{4}}(e'_{(k-1)p+i}) = 0. \tag{5.16}$$

Let us show that

$$u^*(1_A - f_{\frac{1}{4}}(h'))uw^*a = 0. \tag{5.17}$$

By (5.10), one has

$$
\begin{aligned}
(5.18)\qquad & u^*(1_A - f_{\frac{1}{4}}(h'))u \\
= \ & 1_A - u^* \sum_{k=1}^{q} \sum_{i=1}^{p} f_{\frac{1}{4}}(e'_{(k-1)p+i})u \\
= \ & (1_A - f_{\frac{1}{16}}(h')) + f_{\frac{1}{16}}(h') - u^* \sum_{k=1}^{q} \sum_{i=1}^{p} f_{\frac{1}{4}}(e'_{(k-1)p+i})u \\
= \ & (1_A - f_{\frac{1}{16}}(h')) + \sum_{k=1}^{q} f_{\frac{1}{16}}(E_k) - u^* \sum_{k=1}^{q} \sum_{i=1}^{p} f_{\frac{1}{4}}(e'_{(k-1)p+i})u \\
= \ & (1 - f_{\frac{1}{16}}(h')) + u^*(\sum_{k=1}^{q} f_{\frac{1}{16}}(E_k) - \sum_{k=1}^{q} \sum_{i=1}^{p} f_{\frac{1}{4}}(e'_{(k-1)p+i}))u \\
= \ & (1_A - f_{\frac{1}{16}}(h')) + \sum_{k=1}^{q} \sum_{i=1}^{p} u_k^*(f_{\frac{1}{16}}(e'_{(k-1)p+i}) - f_{\frac{1}{4}}(e'_{(k-1)p+i}))u_k \\
= \ & (1_A - f_{\frac{1}{16}}(h')) + \sum_{k=1}^{q} \sum_{i=1}^{p} u_k^* \lambda_{k,i} u_k,
\end{aligned}
$$

where

$$\lambda_{k,i} := f_{\frac{1}{16}}(e'_{(k-1)p+i}) - f_{\frac{1}{4}}(e'_{(k-1)p+i}), \quad k = 1, ..., q,\ i = 1, ..., p.$$

Consider the elements

$$(u_k^* \lambda_{k,i} u_k) w^* a, \quad k = 1, ..., q,\ i = 1, ..., p.$$

If $i = r+1, ..., p$, then, by (5.5),

$$z_k \lambda_{k,i} = \lambda_{k,i} z_k = 0;$$

and hence, by (5.7),

$$[u_k, \lambda_{k,i}] = 0.$$

Since $\lambda_{k,i} \subseteq \mathrm{Her}(1 - f_{\frac{1}{4}}(h'))$, together with (5.12), one has

$$(u_k^* \lambda_{k,i} u_k) w^* a = \lambda_{k,i} w^* a = 0.$$

If $i = 1, ..., r$, then, by (5.7),

$$\lambda_{k,i} u_k w^* a \ = \ \lambda_{k,i} \cos(\frac{\pi}{2}(z_k z_k^* + z_k^* z_k)) w^* a + \lambda_{k,i} z_k^* g(z_k z_k^*) w^* a - \lambda_{k,i} z_k g(z_k^* z_k) w^* a.$$

By (5.5) and (5.14),

$$\lambda_{k,i} z_k^* g(z_k z_k^*) w^* a \in \lambda_{k,i} z_k^* (\overline{cAc}) w^* a = \{0\}.$$

Using (5.14) again, one has $\lambda_{k,i} z_k = 0$, and hence

$$\lambda_{k,i} z_k g(z_k^* z_k) w^* a = 0.$$

Since $[\lambda_{k,i}, z_k^* z_k] = 0$ (by (5.14)),

$$\lambda_{k,i} (\frac{\pi}{2} z_k^* z_k)^{2n} \in \mathrm{Her}(\lambda_{k,i}) \subseteq \mathrm{Her}(1 - f_{\frac{1}{4}}(h')), \quad n = 0, 1, ...,$$

and therefore, by (5.12),

$$\begin{aligned}\lambda_{k,i}\cos(\frac{\pi}{2}(z_kz_k^*+z_k^*z_k))w^*a &= \lambda_{k,i}\sum_{n=0}^{\infty}\frac{1}{(2n)!}(\frac{\pi}{2}(z_kz_k^*+z_k^*z_k))^{2n}w^*a \\ &= \sum_{n=0}^{\infty}\frac{1}{(2n)!}\lambda_{k,i}(\frac{\pi}{2}z_k^*z_k)^{2n}w^*a \\ &= 0.\end{aligned}$$

This shows that

$$\lambda_{k,i}u_kw^*a=0,\quad k=1,...,q,\ i=1,2,...,r,$$

and hence

$$(u_k^*\lambda_{k,i}u_k)w^*a=0,\quad k=1,...,q,\ i=1,...,p. \tag{5.19}$$

Then, together with by (5.18) and (5.12),

$$u^*(1_A-f_{\frac{1}{4}}(h'))uw^*a=(1_A-f_{\frac{1}{16}}(h'))w^*a+\sum_{k=1}^{q}\sum_{i=1}^{p}u_k^*\lambda_{k,i}u_kw^*a=0,$$

and this proves (5.17).

Therefore, with

$$\phi:=f_{\frac{1}{4}}(\phi')\quad\text{and}\quad h:=\phi(1_{pq}),$$

by (5.17), one that,

$$(1_A-h)(uw^*au^*)=0.$$

Set

$$e_i:=\phi(e_{i,i}),\quad i=1,2,...,pq.$$

For any e_i, e_j, $j-i>(l+1)p$, there are $1\leq k_1,k_2\leq q$ with $k_2-k_1\geq l+1$ such that

$$e_i\leq f_{\frac{1}{4}}(E_{k_1})\quad\text{and}\quad e_j\leq f_{\frac{1}{4}}(E_{k_2}).$$

Then it follows from 5.13 that

$$e_i(uw^*au^*)e_j\in\overline{f_{\frac{1}{4}}(E_{k_1})Af_{\frac{1}{4}}(E_{k_1})}uw^*au^*\overline{f_{\frac{1}{4}}(E_{k_2})Af_{\frac{1}{4}}(E_{k_2})}=\{0\}.$$

Set

$$\{i_1,i_2,...,i_{rq}\}=\{(k-1)p+i:k=1,...,q,\ i=1,...,r\}.$$

Then it follows from (5.16) that

$$e_{i_j}(u(w^*a)u^*)=(u(w^*a)u^*)e_{i_j}=0,\quad j=1,2,...,rq,$$

as desired. Finally, it is clear that ϕ is extendable. This proves that ϕ satisfies the assertion, and hence the proposition follows. ☐

Theorem 5.5. *Let A be a unital C*-algebra which has Property (D). Then*

$$\mathrm{ZD}(A)\subseteq\overline{\mathrm{GL}(A)}.$$

In particular, if A is finite, then $A=\overline{\mathrm{GL}(A)}$ (in other words, $\mathrm{tsr}(A)=1$).

Proof. Let $a \in \mathrm{ZD}(A)$, and fix an arbitrary $\varepsilon > 0$ for the time being. Since A has the Property (D), there exist unitaries $u_1, u_2 \in A$ and a $\mathcal{D}_0$-operator $a' \in A$ such that

$$\|u_1 a u_2 - a'\| < \varepsilon.$$

By Proposition 5.4, one has that $a' \in \overline{\mathrm{GL(A)}}$. Since u_1, u_2 are unitaries, it follows that

$$u_1^* a' u_2^* \in \overline{\mathrm{GL}(A)},$$

and hence

$$\mathrm{dist}(a, \overline{\mathrm{GL}(A)}) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, one has that $a \in \overline{\mathrm{GL}(A)}$, and therefore

$$\mathrm{ZD}(A) \subseteq \overline{\mathrm{GL}(A)}. \tag{5.20}$$

If, moreover, the C*-algebra A is finite, by Proposition 3.2 of [22],

$$A \setminus \mathrm{GL}(A) \subseteq \overline{\mathrm{ZD}(A)};$$

and then, together with (5.20),

$$A = \mathrm{GL}(A) \cup (A \setminus \mathrm{GL}(A)) \subseteq \mathrm{GL}(A) \cup \overline{\mathrm{ZD}(A)} \subseteq \mathrm{GL}(A) \cup \overline{\mathrm{GL}(A)} = \overline{\mathrm{GL}(A)},$$

as desired. □

6. Non-invertible elements and zero divisors of $\mathrm{C}(X) \rtimes \Gamma$

In the following two sections, let us show that the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ has Property (D) if (X, Γ) has the (URP) and (COS). Since $\mathrm{C}(X) \rtimes \Gamma$ is finite, this shows that $\mathrm{C}(X) \rtimes \Gamma$ has stable rank one by Theorem 5.5.

Recall that if B is a sub-C*-algebra of A, a conditional expectation from A to B is a completely positive linear contraction $\mathbb{E} : A \to B$ such that

$$\mathbb{E}(b) = b, \quad \mathbb{E}(ba) = b\mathbb{E}(a) \quad \text{and} \quad \mathbb{E}(ab) = \mathbb{E}(a)b, \quad b \in B,\ a \in A.$$

If (X, Γ) is a free dynamical system, and if $\mathbb{E} : \mathrm{C}(X) \rtimes \Gamma \to \mathrm{C}(X)$ is a conditional expectation, where $\mathrm{C}(X) \rtimes \Gamma$ is a crossed-product C*-algebra. Then

$$\mathbb{E}(u_\gamma) = 0, \quad \gamma \in \Gamma \setminus \{e\}.$$

Indeed, let $\gamma \in \Gamma \setminus \{e\}$ and consider $\mathbb{E}(u_\gamma) \in \mathrm{C}(X)$. Note that for all $g \in \mathrm{C}(X)$, since $u_\gamma^* g u_\gamma \in \mathrm{C}(X)$, one has

$$g\mathbb{E}(u_\gamma) = \mathbb{E}(g u_\gamma) = \mathbb{E}(u_\gamma u_\gamma^* g u_\gamma) = \mathbb{E}(u_\gamma)(u_\gamma^* g u_\gamma).$$

Assume $\mathbb{E}(u_\gamma) \neq 0$. Then there is $x_0 \in X$ such that $\mathbb{E}(u_\gamma)(x_0) \neq 0$. Since (X, Γ) is free, one has $x_0 \neq x_0\gamma^{-1}$. Pick $g \in \mathrm{C}(X)$ such that $g(x_0) = 1$ and $g(x_0\gamma^{-1}) = 0$. Then

$$g(x_0)\mathbb{E}(u_\gamma)(x_0) = \mathbb{E}(u_\gamma)(x_0) \neq 0 = \mathbb{E}(u_\gamma)(x_0) g(x_0\gamma^{-1}) = \mathbb{E}(u_\gamma)(x_0)(u_\gamma^* g u_\gamma)(x_0),$$

which is a contradiction.

If Γ is amenable, then a conditional expectation $\mathbb{E} : \mathrm{C}(X) \rtimes \Gamma \to \mathrm{C}(X)$ always exists, and is not only unique (see above) but also faithful (see, for instance, Proposition 4.1.9 of [4]).

Lemma 6.1. *Let (X, Γ) be a free and minimal topological dynamical system, where Γ is a countable discrete group and X is a compact Hausdorff space. Denote by $A = \mathrm{C}(X) \rtimes \Gamma$ a crossed-product C*-algebra, and assume there is a faithful conditional expectation $\mathbb{E} : A \to \mathrm{C}(X)$.*

Let $a \in A$ such that $ba = 0$ for some non-zero positive element b. Then, for any $\varepsilon > 0$, there is unitary $u \in A$ and a (non-empty) open set $E \subseteq X$ such that

$$\|\varphi_E ua\| < \varepsilon.$$

Proof. Since $\mathbb{E}$ is faithful, without loss of generality, one may assume

$$\|a\| = 1 \quad \text{and} \quad \|\mathbb{E}(b)\| = 1.$$

Pick $\varepsilon' > 0$ such that if $\|ca\| < \varepsilon'$ for some positive element c with $\|c\| \leq (\|b\| + 1)^2$, then $\left\|c^{\frac{1}{2}}a\right\| < \varepsilon/(\|b\| + 1)$. Pick $\varepsilon'' \in (0, 1)$ such that

$$(\|b\| + \varepsilon'')\varepsilon'' < \varepsilon',$$

and pick $b' \in \mathrm{C_c}(\Gamma, \mathrm{C}(X))$ such that

$$\|b - b'\| < \varepsilon''.$$

Since $\|\mathbb{E}(b)\| = 1$, one may assume that $\|\mathbb{E}(b')\| = 1$.

Write

$$b' = \sum_{\gamma \in \Gamma_0} f_\gamma u_\gamma$$

for a finite set $\Gamma_0 \subseteq \Gamma$ with $\Gamma_0 = \Gamma_0^{-1}$, where $f_\gamma \in \mathrm{C}(X)$. Since $\mathbb{E}(b') = f_e$, one has that $\|f_e\| = 1$, and then there is $x_0 \in X$ such that

$$|f_e(x_0)| = 1.$$

Pick a neighbourhood U of x_0 such that $\overline{\bigcup_{\gamma \in \Gamma_0} U\gamma} \neq X$, and pick an open set $W \neq X$ such that

$$\overline{\bigcup_{\gamma \in \Gamma_0} U\gamma} \subseteq W.$$

Therefore, there is a continuous function $\varphi_W : X \to [0, 1]$ such that

$$\varphi_W^{-1}((0, 1]) = W \quad \text{and} \quad \bigcup_{\gamma \in \Gamma_0} U\gamma \subseteq \varphi_W^{-1}(1). \tag{6.1}$$

Pick a continuous function $\varphi_U : X \to [0, 1]$ so that

$$\varphi_U^{-1}((0, 1]) = U \quad \text{and} \quad \varphi_U(x_0) = 1. \tag{6.2}$$

Note that

$$\begin{aligned} b'\varphi_U(b')^* &= (\sum_{\gamma\in\Gamma_0} f_\gamma u_\gamma)\varphi_U(\sum_{\gamma\in\Gamma_0} f_\gamma u_\gamma)^* \\ &= (\sum_{\gamma\in\Gamma_0} f_\gamma u_\gamma)\varphi_U(\sum_{\gamma\in\Gamma_0} u_\gamma^*\overline{f_\gamma}) \\ &= \sum_{\gamma,\gamma'\in\Gamma_0} f_{\gamma'}u_{\gamma'}\varphi_U u_\gamma^*\overline{f_\gamma} \\ &= \sum_{\gamma,\gamma'\in\Gamma_0} f_{\gamma'}(\overline{f_\gamma}\circ(\gamma'\gamma^{-1}))(\varphi_U\circ\gamma')u_{\gamma'\gamma^{-1}}. \end{aligned}$$

Hence, by (6.1),

$$\varphi_W(b'\varphi_U(b')^*) = b'\varphi_U(b')^*. \tag{6.3}$$

Also note that, by (6.2),

$$\mathbb{E}(b'\varphi_U(b')^*)(x_0) = \sum_{\gamma\in\Gamma_0} |f_\gamma(x_0)|^2\,\varphi_U(x_0\gamma) \geq |f_e(x_0)|^2 = 1;$$

and in particular,

$$\|\mathbb{E}(b'\varphi_U(b')^*)\| \geq 1. \tag{6.4}$$

Set

$$b'' = \frac{1}{\|\mathbb{E}(b'\varphi_U(b')^*)\|}b'\varphi_U(b')^*.$$

Note that

$$\mathbb{E}(b'') = \frac{1}{\left\|\sum_{\gamma\in\Gamma_0}|f_\gamma|^2\varphi_U\right\|}\sum_{\gamma\in\Gamma_0}|f_\gamma|^2\,\varphi_{U\gamma}.$$

So there is $y_0\in X$ so that $\mathbb{E}(b'')(y_0)=1$. By perturbing f_γ, $\gamma\in\Gamma_0$, and φ_U to be locally constant around y_0, there is an open neighbourhood $V\ni y_0$ such that

$$\mathbb{E}(b'')(x) = \mathbb{E}(b'')(y_0) = 1, \quad x\in V. \tag{6.5}$$

Moreover, since (X,Γ) is free, one may choose V small enough so that

$$V\cap V\gamma = \varnothing, \quad \gamma\in\Gamma_0^2\setminus\{e\}, \tag{6.6}$$

and since the action is minimal (so any orbit is dense), by choosing V even smaller, there is $\gamma_0\in\Gamma$ such that

$$V\gamma_0\cap W = \varnothing. \tag{6.7}$$

Note that, by (6.4) and since b is positive,

$$\begin{aligned} \|b''a\| &= \left\|\frac{1}{\|\mathbb{E}(b'\varphi_U(b')^*)\|}b'\varphi_U(b')^*a\right\| \\ &= \frac{1}{\|\mathbb{E}(b'\varphi_U(b')^*)\|}\|b'\varphi_U(b')^*a\| \\ &\approx_{(\|b\|+\varepsilon'')\varepsilon''} \frac{1}{\|\mathbb{E}(b'\varphi_U(b')^*)\|}\|b'\varphi_U ba\| = 0. \end{aligned}$$

That is,

$$\|b''a\| \leq (\|b\| + \varepsilon'')\varepsilon'' < \varepsilon'. \tag{6.8}$$

Since

$$\|b''\| \leq \|b'\|^2 \leq (\|b\| + 1)^2, \tag{6.9}$$

by the choice of ε', one has

$$\left\|(b'')^{\frac{1}{2}}a\right\| < \frac{\varepsilon}{\|b\| + 1}. \tag{6.10}$$

Now, choose a continuous function $h : X \to [0, 1]$ such that $h^{-1}((0, 1]) \subseteq V$, and $f^{-1}(1)$ contains a neighbourhood of y_0. Note that $\|h\| = 1$.

By (6.6),

$$hu_\gamma h = 0, \quad \gamma \in \Gamma_0^2 \setminus \{e\},$$

and hence, writing

$$b'' = \sum_{\gamma \in \Gamma_0^2} c_\gamma u_\gamma,$$

together with (6.5), one has

$$hb''h = h(\sum_{\gamma \in \Gamma_0^2} c_\gamma u_\gamma)h = \sum_{\gamma \in \Gamma_0^2} c_\gamma hu_\gamma h = c_e h^2 = \mathbb{E}(b'')h^2 = h^2. \tag{6.11}$$

By (6.7), one has $u_{\gamma_0} h u_{\gamma_0}^* \perp \varphi_W$ and hence, by (6.3),

$$u_{\gamma_0} h u_{\gamma_0}^* \perp b''. \tag{6.12}$$

Consider

$$v := u_{\gamma_0} h (b'')^{\frac{1}{2}}.$$

Then

$$vv^* = u_{\gamma_0} hb''h u_{\gamma_0}^* = u_{\gamma_0} h^2 u_{\gamma_0}^*$$

and

$$v^*v = (b'')^{\frac{1}{2}} h^2 (b'')^{\frac{1}{2}}.$$

Pick an open set E such that

$$\varphi_E vv^* = \varphi_E(u_{\gamma_0} h^2 u_{\gamma_0}^*) = \varphi_E. \tag{6.13}$$

(Such E exists because h is constantly 1 in a small neighbourhood of y_0.)

By (6.12), $vv^* \perp v^*v$. By Lemma 3.2 and (6.13), there is a unitary $u \in A$ such that

$$(u^*\varphi_E u)(b'')^{\frac{1}{2}} h^2 (b'')^{\frac{1}{2}} = (u^*\varphi_E u)v^*v = u^*\varphi_E u.$$

Therefore, by (6.9), (6.10),

$$\begin{aligned}
\|\varphi_E u a\| &= \|u(u^*\varphi_E u)a\| = \left\|u(u^*\varphi_E u)(b'')^{\frac{1}{2}} h^2 (b'')^{\frac{1}{2}} a\right\| \\
&\leq (\|b\| + 1)\left\|(b'')^{\frac{1}{2}} a\right\| \\
&< \varepsilon,
\end{aligned}$$

as desired. $\square$

Proposition 6.2. *Let (X, Γ) be a free and minimal topological dynamical system, where Γ is a countable discrete group and X is a compact Hausdorff space. Denote by $A = \mathrm{C}(X) \rtimes \Gamma$ a crossed-product C*-algebra. Assume A is finite and there is a faithful conditional expectation $\mathbb{E} : A \to \mathrm{C}(X)$.*

Let $a \in A$ be a non-invertible element. Then, for any $\varepsilon > 0$, there exist $b \in \mathrm{C_c}(\Gamma, \mathrm{C}(X))$, a (non-empty) open set $E \subseteq X$, and unitaries $u_1, u_2 \in A$ such that

$$\|u_1 a u_2 - b\| < \varepsilon \quad \textit{and} \quad \varphi_E b = b \varphi_E = 0.$$

Proof. Without loss of generality, one may assume that $\|a\| = 1$. Let $\varepsilon > 0$ be given. Since a is not invertible and A is finite, by Proposition 3.2 of [22], there are $a' \in A$ and nonzero $b_1, b_2 \in A^+$ such that

$$\|a'\| = 1, \quad \|a - a'\| < \varepsilon/5, \quad \text{and} \quad b_1 a' = 0 = a' b_2.$$

By Lemma 6.1, there are unitaries $u_1, u_2 \in A$ and open sets $E', F' \subseteq X$ such that

$$\|\varphi_{E'} u_1 a'\| < \varepsilon/5 \quad \text{and} \quad \|a' u_2 \varphi_{F'}\| < \varepsilon/5.$$

Since (X, Γ) is minimal (so any orbit is dense), by passing to smaller open sets and changing the unitary u_2, one may assume that $E' = F'$.

Pick $a'' \in \mathrm{C_c}(\Gamma, \mathrm{C}(X))$ such that

$$\|u_1 a' u_2 - a''\| < \varepsilon/5,$$

and note that

$$\begin{aligned} u_1 a u_2 &\approx_{\varepsilon/5} u_1 a' u_2 \\ &= \varphi_{E'} u_1 a' u_2 \varphi_{E'} + (1 - \varphi_{E'}) u_1 a' u_2 \varphi_{E'} + \\ &\quad\ \varphi_{E'} u_1 a' u_2 (1 - \varphi_{E'}) + (1 - \varphi_{E'}) u_1 a' u_2 (1 - \varphi_{E'}) \\ &\approx_{3\varepsilon/5} (1 - \varphi_{E'}) u_1 a' u_2 (1 - \varphi_{E'}) \\ &\approx_{\varepsilon/5} (1 - \varphi_{E'}) a'' (1 - \varphi_{E'}) \end{aligned}$$

Pick an open set $E \subseteq \varphi_{E'}^{-1}(1)$ (so that $\varphi_E \varphi_{E'} = \varphi_E$), and define

$$b = (1 - \varphi_{E'}) a'' (1 - \varphi_{E'}).$$

Then it is clear that

$$\|u_1 a u_2 - b\| < \varepsilon \quad \text{and} \quad \varphi_E b = b \varphi_E = 0,$$

as desired. □

7. Stable rank of $\mathrm{C}(X) \rtimes \Gamma$

In this section, assuming (X, Γ) has the (URP) and (COS), let us show that the element b obtained in Proposition 6.2 is an $\mathcal{D}_0$-operator (Proposition 7.7). Hence the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ has Property (D), and has stable rank one by Theorem 5.5.

Let Γ be a discrete amenable group, and let $\Gamma_1, \Gamma_2, ..., \Gamma_T$ be finite subsets of Γ. Recall that Γ is said to be tiled by $\Gamma_1, \Gamma_2, ..., \Gamma_T$ if there are sets group elements

$$\gamma_{i,n}, \quad n = 1, 2, ..., \ i = 1, ..., T,$$

such that

$$\Gamma = \bigsqcup_{i=1}^{T} \bigsqcup_{n=1}^{\infty} \gamma_{i,n}\Gamma_i.$$

Note that if Γ_i is (right) $(\mathcal{F}, \varepsilon)$-invariant, then its (left) translation $\gamma_{i,n}\Gamma_i$ is also $(\mathcal{F}, \varepsilon)$-invariant.

Lemma 7.1. *Let Γ be an infinite amenable group, and let $\Gamma_1, \Gamma_2, ..., \Gamma_T \subseteq \Gamma$ be finite sets which tile Γ. Let $\delta \in (0, 1]$ and let $n \in \mathbb{N}$. Then, there is $(\mathcal{F}, \varepsilon)$ such that if $F \subseteq \Gamma$ is $(\mathcal{F}, \varepsilon)$-invariant, then there is $H \subseteq F$ such that H is tiled by $\Gamma_1, \Gamma_2, ..., \Gamma_T$ with multiplicities divided by n, and*

$$\frac{|H|}{|F|} > 1 - \delta.$$

Proof. Set $K = \Gamma_1 \cup \cdots \cup \Gamma_T$, and choose $\delta' > 0$ such that

$$(1 - \delta')(1 - \frac{\delta'}{2}) > \delta.$$

Choose $(\mathcal{F}, \varepsilon)$ sufficiently large such that if F is $(\mathcal{F}, \varepsilon)$-invariant, then

$$\frac{|\mathrm{int}_K(F)|}{|F|} > 1 - \frac{\delta'}{2}. \tag{7.1}$$

Since Γ is infinite, one may assume that $(\mathcal{F}, \varepsilon)$ large enough so that if F is $(\mathcal{F}, \varepsilon)$-invariant, then

$$|F| > \frac{2n(|\Gamma_1| + \cdots + |\Gamma_T|)}{(2 - \delta')\delta'}. \tag{7.2}$$

Then this $(\mathcal{F}, \varepsilon)$ satisfies the property of the lemma.

Indeed, let F be an $(\mathcal{F}, \varepsilon)$-invariant set. Since $\Gamma_1, ..., \Gamma_T$ tile Γ, by (7.1), there is a set $F' \subseteq F$ such that F' can be tiled by $\Gamma_1, ..., \Gamma_T$ (in fact, F' can be chosen as as the union of the tiles which intersect with $\mathrm{int}_K F$) and

$$\frac{|F'|}{|F|} > 1 - \frac{\delta'}{2}. \tag{7.3}$$

Write

$$F' = (\bigsqcup_{i=1}^{m_1} \gamma_{1,i}\Gamma_1) \sqcup \cdots \sqcup (\bigsqcup_{i=1}^{m_T} \gamma_{T,i}\Gamma_T),$$

where $\gamma_{i,j} \in \Gamma$ and m_i, $i = 1, 2, ..., T$, are non-negative integers. Note that

$$m_1 |\Gamma_1| + \cdots + m_T |\Gamma_T| = |F'|.$$

For each m_i, $i = 1, 2, ..., T$, consider r_i which is the remainder of m_i divided by n. Then, set

$$H = (\bigsqcup_{i=1}^{m_1 - r_1} \gamma_{1,m_1}\Gamma_1) \sqcup \cdots \sqcup (\bigsqcup_{i=1}^{m_T - r_T} \gamma_{T,i}\Gamma_T).$$

It is clear that H is tiled by $\Gamma_1, \Gamma_2, ..., \Gamma_T$ with multiplicities divided by n. Moreover, by (7.3) and (7.2),

$$\begin{aligned} 1 - \frac{|H|}{|F'|} &= \frac{r_1 |\Gamma_1| + \cdots + r_T |\Gamma_T|}{|F'|} < \frac{n |\Gamma_1| + \cdots + n |\Gamma_T|}{|F'|} \\ &< \frac{2n(|\Gamma_1| + \cdots + |\Gamma_T|)}{(2-\delta')} \frac{1}{|F|} < \delta', \end{aligned}$$

and hence, by (7.3) again,

$$\frac{|H|}{|F|} > (1-\delta')(1-\frac{\delta'}{2}) > 1-\delta,$$

as desired. $\square$

Lemma 7.2. *Let Γ be an infinite amenable group, and let $\Gamma_1, \Gamma_2, ..., \Gamma_T \subseteq \Gamma$ be finite sets which tile Γ. Let $\delta \in (0, 1]$, let $n \in \mathbb{N}$, and let $K \subseteq \Gamma$ be a finite set. Then, there exists $(\mathcal{F}, \varepsilon)$ such that if*

$$F_1, F_2, ..., F_n$$

are mutually disjoint $(\mathcal{F}, \varepsilon)$-invariant sets and

$$|F_1| = |F_2| = \cdots = |F_n|,$$

then there are $H_1 \subseteq F_1, ..., H_n \subseteq F_n$ such that

$$H_iK \subseteq F_i, \quad i = 1, 2, ..., n,$$

each H_i is tiled by $\Gamma_1, \Gamma_2, ..., \Gamma_T$ with multiplicities divided by n,

$$|H_1| = |H_2| = \cdots = |H_n|,$$

and

$$\frac{|H_i|}{|F_i|} > 1 - \delta, \quad i = 1, 2, ..., n.$$

Proof. Apply Lemma 7.1 to

$$\frac{\delta}{2(1+T)} \quad \text{and} \quad n |\Gamma_1| |\Gamma_2| \cdots |\Gamma_T|,$$

one obtains $(\mathcal{F}', \varepsilon')$. Choose $(\mathcal{F}, \varepsilon)$ such that if F is $(\mathcal{F}, \varepsilon)$-invariant, then $\mathrm{int}_K F$ is $(\mathcal{F}', \varepsilon')$-invariant, and

$$\frac{|\mathrm{int}_K F|}{|F|} > 1 - \frac{\delta}{2(1+T)}. \tag{7.4}$$

Then $(\mathcal{F}, \varepsilon)$ satisfies the property of the lemma.

Indeed, let $F_1, F_2, ..., F_n$ be mutually disjoint $(\mathcal{F}, \varepsilon)$-invariant sets with

$$|F_1| = |F_2| = \cdots = |F_n|.$$

Consider the sets

$$\mathrm{int}_K F_1, \ \mathrm{int}_K F_2, \ ..., \mathrm{int}_K F_n.$$

Then each of them is $(\mathcal{F}', \varepsilon')$-invariant. Also note that

$$(\mathrm{int}_K F_i)K \subseteq F_i, \quad (\mathrm{int}_K F_i)K \cap (\mathrm{int}_K F_j) = \varnothing, \quad i, j = 1, 2, ..., n, \ i \neq j,$$

By Lemma 7.1, there are

$$F_1' \subseteq \mathrm{int}_K F_1, \ F_2' \subseteq \mathrm{int}_K F_2, ..., \ F_n' \subseteq \mathrm{int}_K F_n,$$

such that

$$\frac{|F_i'|}{|\mathrm{int}_K F_i|} > 1 - \frac{\delta}{2(1+T)}, \quad i = 1, 2, ..., n, \tag{7.5}$$

and

$$F_i' = (\bigsqcup_{j=1}^{m_1^{(i)}} \gamma_{1,j}^{(i)} \Gamma_1) \sqcup \cdots \sqcup (\bigsqcup_{j=1}^{m_T^{(i)}} \gamma_{T,j}^{(i)} \Gamma_T), \quad i = 1, 2, ..., n, \tag{7.6}$$

and each $m_t^{(i)}$, $i = 1, 2, ..., n$, $t = 1, 2, ..., T$, is divided by $n\,|\Gamma_1|\,|\Gamma_2| \cdots |\Gamma_T|$.

It follows from (7.4) and (7.5) that for each $i = 1, 2, ..., n$,

$$\begin{aligned} |F_i| - |F_i'| &= (|F_i| - |\mathrm{int}_K(F_i)|) + (|\mathrm{int}_K(F_i)| - |F_i'|) \\ &< \frac{\delta}{2(1+T)}\,|F_i| + \frac{\delta}{2(1+T)}\,|\mathrm{int}_K F_i| \\ &\leq \frac{\delta}{2(1+T)}\,|F_i| + \frac{\delta}{2(1+T)}\,|F_i| \\ &= \frac{\delta}{1+T}\,|F_i|\,. \end{aligned} \tag{7.7}$$

In the decomposition (7.6), if

$$m_t^{(i)} < \frac{\delta}{1+T}\,|F_i|\,,$$

then set $m_t^{(i)} = 0$, and denote this possibly smaller new sets still by F_i'. Then, by (7.7), with the set new F_i', one has

$$0 \leq |F_i| - |F_i'| < \frac{\delta}{1+T}\,|F_i| + T\frac{\delta}{1+T}\,|F_i| = \delta\,|F_i|\,, \quad i = 1, 2, ..., n. \tag{7.8}$$

Also note that if $m_t^{(i)} \neq 0$, then

$$m_t^{(i)} \geq \frac{\delta}{1+T}\,|F_i| \geq \delta\,|F_i|\,. \tag{7.9}$$

Set

$$D = \min\{|F_1'|\,, |F_2'|\,, ..., |F_n'|\}.$$

Since $|F_1'|\,, |F_2'|\,, ..., |F_n'|$ are divided by $n\,|\Gamma_1|\,|\Gamma_2| \cdots |\Gamma_T|$, there are non-negative integers d_i, $i = 1, ..., n$, such that

$$|F_i'| - D = d_i\,|\Gamma_1|\,|\Gamma_2| \cdots |\Gamma_T|\,n, \quad i = 1, 2, ..., n.$$

By (7.8) (and note that $|F_1| = \cdots = |F_n|$),

$$\frac{D}{|F_i|} > \frac{|F_i| - \delta\,|F_i|}{|F_i|} = 1 - \delta, \quad i = 1, 2, ..., n, \tag{7.10}$$

and so

$$|F_i'| - D \leq |F_i| - D \leq \delta\,|F_i|\,, \quad i = 1, 2, ..., n.$$

For each $i = 1, 2, ..., n$, consider

$$\{t_1, t_2, ..., t_S\} = \left\{t = 1, 2, ..., T : m_t^{(i)} \neq 0\right\}.$$

Then, there are

$$0 \leq c_{t_1}^{(i)}, ..., c_{t_S}^{(i)} \leq d_i\,|\Gamma_1|\,|\Gamma_2| \cdots |\Gamma_T| \leq \frac{\delta\,|F_i|}{n}$$

such that

$$d_i\,|\Gamma_1|\,|\Gamma_2| \cdots |\Gamma_T| = c_{t_1}^{(i)}\,|\Gamma_{t_1}| + \cdots + c_{t_S}^{(i)}\,|\Gamma_{t_S}|\,.$$

(Actually, one can choose $c_{t_1}^{(i)} = d_i\,|\Gamma_1|\,|\Gamma_2| \cdots |\Gamma_T|\,/\,|\Gamma_{t_1}|$ and $c_{t_s}^{(i)} = 0$, $s = 2, ..., S$.) Note that, by (7.9),

$$\frac{\delta\,|F_i|}{n} \leq \frac{m_{t_s}^{(i)}}{n}, \quad s = 1, 2, ..., S.$$

For each $t \notin \{t_1, ..., t_S\}$, set $c_t^{(i)} = 0$. Then, one has that

$$0 \leq c_t^{(i)} \leq d_i\,|\Gamma_1|\,|\Gamma_2| \cdots |\Gamma_T| \leq \frac{\delta\,|F_i|}{n} \leq \frac{m_t^{(i)}}{n}, \quad t = 1, 2, ..., T, \tag{7.11}$$

and

$$d_i\,|\Gamma_1|\,|\Gamma_2| \cdots |\Gamma_T| = c_1^{(i)}\,|\Gamma_1| + \cdots + c_T^{(i)}\,|\Gamma_T|\,.$$

Put

$$H_i = \Big(\bigsqcup_{j=1}^{m_1^{(i)} - c_1^{(i)} n} \gamma_{1,j}^{(i)}\Gamma_1\Big) \sqcup \cdots \sqcup \Big(\bigsqcup_{j=1}^{m_T^{(i)} - c_T^{(i)} n} \gamma_{T,j}^{(i)}\Gamma_T\Big), \quad i = 1, 2, ..., n.$$

(Note that, by (7.11), $m_t^{(i)} - c_t^{(i)} n \geq 0$.) Since each $m_t^{(i)}$ is divisible by n, it is clear that each H_i is tiled by $\Gamma_1, ..., \Gamma_T$ with multiplicities divisible by n. Since

$$H_1 \subseteq \mathrm{int}_K F_1,\ H_2 \subseteq \mathrm{int}_K F_2, ...,\ H_n \subseteq \mathrm{int}_K F_n,$$

one has

$$H_1 K \subseteq F_1,\ H_2 K \subseteq F_2, ...,\ H_n K \subseteq F_n,$$

Also note that

$$|H_1| = |H_2| = \cdots = |H_n| = D,$$

and hence, by (7.10),

$$\frac{|H_i|}{|F_i|} > 1 - \delta, \quad i = 1, 2, ..., n,$$

as desired. □

Lemma 7.3. *Let Γ be an infinite amenable group, and let (X, Γ) be a minimal dynamical system with the (URP). Let $\lambda > 0$ be arbitrary, and let $O_{0,1}, ..., O_{0,M}, O_{1,1}, ..., O_{1,M} \subseteq X$ be mutually disjoint non-empty open sets together with*

$$\{\kappa_{0,1}(= e), \kappa_{0,2}, ..., \kappa_{0,M}, \kappa_{1,1}(= e), \kappa_{1,2}, ..., \kappa_{1,M}\} \subseteq \Gamma$$

such that

$$O_{i,m} = O_{i,1}\kappa_{i,m}, \quad i = 0, 1, \ m = 1, ..., M.$$

Put

$$\delta := \min\{\mu(O_{i,m}) : i = 0, 1, \ m = 1, ..., M, \ \mu \in \mathcal{M}_1(X, \Gamma)\},$$

and let $K \subseteq \Gamma$ be a symmetric finite set. (Since (X, Γ) is minimal, one has that $\delta > 0$.)

Then, there are $(\mathcal{F}, \varepsilon)$, and $n \in \mathbb{N}$ ($n > 3$) such that if (B, F) is a tower of (X, Γ) with F being $(\mathcal{F}, \varepsilon)$-invariant, then there is an order zero c.p.c. map

$$\phi : \mathrm{M}_{n^2}(\mathbb{C}) \to A,$$

where $A = \mathrm{C}(X) \rtimes \Gamma$, such that if

$$h = \phi(1) \quad \text{and} \quad e_i = \phi(e_{i,i}), \quad i = 1, 2, ..., n^2,$$

and

$$b_k := e_{n(k-1)+1} + \cdots + e_{n(k-1)+n}, \quad k = 1, 2, ..., n,$$

then

$$e_i \in \mathrm{C}(X)$$

and if denote by

$$E_i = e_i^{-1}((0, 1]), \quad i = 1, 2, ..., n^2,$$

then

(1)
$$\bigsqcup_{i=1}^{n^2} E_i \subseteq \bigsqcup_{\gamma \in F} B\gamma,$$
and
$$\mu(\bigsqcup_{\gamma\in F} B\gamma \setminus \bigsqcup_{i=1}^{n^2} E_i) < \lambda\frac{\delta}{16n}\mu(\bigsqcup_{\gamma\in F} B\gamma), \quad \mu \in \mathcal{M}_1(X, \Gamma).$$

(2) *for each $k = 1, 2, ..., n$, there are mutually disjoint open sets $O^k_{0,1}, ..., O^k_{0,M}$ and $O^k_{1,1}, ..., O^k_{1,M}$ such that*
 (a) $O^k_{0,m} \subseteq O_{0,m} \cap \bigsqcup_{j=n(k-1)+4}^{nk} E_j$ *and* $O^k_{1,m} \subseteq O_{1,m} \cap \bigsqcup_{j=n(k-1)+1}^{nk} E_j$, $m = 1, 2, ..., M$,
 (b) $O^k_{i,m} = O^k_{i,1}\kappa_{i,m}$, $i = 0, 1$, $m = 1, 2, ..., M$,
 (c) $\mu(O^k_{0,1}), \mu(O^k_{1,1}) > \frac{\delta}{8n}\mu(\bigsqcup_{i=1}^{n^2} E_i)$, $\mu \in \mathcal{M}_1(X, \Gamma)$.

(3) $b_{k_1} \perp u_\gamma b_{k_2} u^*_\gamma$, $\quad \gamma \in K$, $k_1 \neq k_2$, $1 \leq k_1, k_2 \leq n$, *where $u_\gamma \in A$ is the canonical unitary of γ.*

Moreover, n can be chosen arbitrarily large.

Proof. Choose $n \in \mathbb{N}$ such that

$$0 < \frac{1}{n-3} < \frac{\delta}{24}, \quad \lambda\frac{\delta}{16n} < \frac{1}{2}, \quad \text{and} \quad \frac{3}{n} < \frac{\delta}{16}. \tag{7.12}$$

Pick $(\mathcal{F}', \varepsilon')$ such that if a finite set $\Gamma_0 \subseteq \Gamma$ is $(\mathcal{F}', \varepsilon')$-invariant, then

$$\frac{1}{|\Gamma_0|}\,|\{\gamma \in \Gamma_0 : x\gamma \in O_{i,m}\}| > \frac{\delta}{2}, \quad x \in X, \ i = 0, 1, \ m = 0, 1, ..., M, \tag{7.13}$$

and

$$\frac{\left|\partial_{K_0^M}\Gamma_0\right|}{|\Gamma_0|} < \frac{\delta}{16}, \tag{7.14}$$

where

$$K_0 := \{\kappa_{0,1}, \kappa_{0,2}, ..., \kappa_{0,M}, \kappa_{1,1}, \kappa_{1,2}, ..., \kappa_{1,M}\}.$$

By Theorem 4.3 of [6], there are $(\mathcal{F}', \varepsilon')$-invariant finite sets

$$\Gamma_1,\ \Gamma_2,\ ...,\ \Gamma_T \subseteq \Gamma$$

which tile Γ. Applying Lemma 7.2 to $\lambda\delta/32n$, n, and K with respect to the finite sets Γ_1,..., Γ_T, one obtains $(\mathcal{F}'', \varepsilon'')$.

By Theorem 4.3 of [6] again, there are $(\mathcal{F}'', \varepsilon'')$-invariant finite sets

$$\Gamma_1',\ \Gamma_2',\ ...,\ \Gamma_{T'}' \subseteq \Gamma$$

which tile Γ. Applying Lemma 7.1 to $\lambda\delta/32n$ and n with respect to the finite sets Γ_1', Γ_2', ..., $\Gamma_{T'}'$, one obtains $(\mathcal{F}, \varepsilon)$. Since Γ is infinite, one may assume that $(\mathcal{F}, \varepsilon)$ is sufficiently large such that if F is $(\mathcal{F}, \varepsilon)$-invariant, then $|F| > 2n^2$.

Then, $(\mathcal{F}, \varepsilon)$ satisfies the property of the lemma.

Indeed, let (B, F) be a tower such that F is $(\mathcal{F}, \varepsilon)$-invariant. Then, by Lemma 7.1, there is a finite set $R_1 \subseteq F$ such that

$$\frac{|R_1|}{|F|} < \lambda\frac{\delta}{32n} \tag{7.15}$$

and $F \setminus R_1$ can be tiled by $\Gamma_1', ..., \Gamma_{T'}'$ with multiplicities divided by n. By a grouping of the tilings, one has

$$F \setminus R_1 = \Gamma_1'' \sqcup \Gamma_2'' \sqcup \cdots \sqcup \Gamma_n'',$$

where Γ_i'', $i = 1, ..., n$, are mutually disjoint $(\mathcal{F}'', \varepsilon'')$-invariant set and

$$|\Gamma_1''| = |\Gamma_2''| = \cdots = |\Gamma_n''|.$$

By Lemma 7.2 and the choice of $(\mathcal{F}'', \varepsilon'')$, there are finite sets $\Gamma_i''' \subseteq \Gamma_i''$ such that

$$\Gamma_i'''K \subseteq \Gamma_i'', \quad i = 1, 2, ..., n,$$

$$\frac{|\Gamma_i'''|}{|\Gamma_i''|} > 1 - \lambda\frac{\delta}{32n}, \quad i = 1, 2, ..., n,$$

$$|\Gamma_1'''| = |\Gamma_2'''| = \cdots = |\Gamma_n'''|,$$

and each Γ_i''' is tiled by $\Gamma_1, ..., \Gamma_T$ with multiplicities divided by n. Since Γ_i'', $i = 1, ..., n$, are mutually disjoint, one has

$$\Gamma_i'''K \cap \Gamma_j''' = \varnothing, \quad i, j = 1, 2, ..., n,\ i \neq j.$$

Write

$$R_2 = (F \setminus R_1) \setminus (\Gamma_1''' \sqcup \Gamma_2''' \sqcup \cdots \sqcup \Gamma_n'''),$$

and one has

$$F \setminus (R_1 \cup R_2) = \Gamma_1''' \sqcup \Gamma_2''' \sqcup \cdots \sqcup \Gamma_n''',$$

and

$$\frac{|R_2|}{|F|} \leq \lambda \frac{\delta}{32n}. \tag{7.16}$$

Note that, by (7.12), (7.15), and (7.16), one has

$$2\,|F \setminus (R_1 \cup R_2)| > |F|\,. \tag{7.17}$$

Then, inside each Γ_i''', since it is tiled by $\Gamma_1, ..., \Gamma_T$ (which are $(\mathcal{F}', \varepsilon')$-invariant) with multiplicities divided by n, after a grouping, one has

$$\Gamma_i''' = \Gamma_{i,1} \sqcup \cdots \sqcup \Gamma_{i,n},$$

where $\Gamma_{i,j}$ is $(\mathcal{F}', \varepsilon')$-invariant with

$$|\Gamma_{i,1}| = |\Gamma_{i,2}| = \cdots = |\Gamma_{i,n}|\,, \quad i = 1, 2, ..., n.$$

In summary, one obtains the decomposition

$$F \setminus (R_1 \cup R_2) = (\Gamma_{1,1} \sqcup \cdots \sqcup \Gamma_{1,n}) \sqcup \cdots \sqcup (\Gamma_{n,1} \sqcup \cdots \sqcup \Gamma_{n,n})$$

with properties

(1)
$$|\Gamma_{i_1,j_1}| = |\Gamma_{i_2,j_2}|\,, \quad 1 \leq i_1, i_2, j_1, j_2 \leq n, \tag{7.18}$$

(2) each $\Gamma_{i,j}$ is $(\mathcal{F}', \varepsilon')$-invariant,

(3)
$$(\Gamma_{i,1} \sqcup \cdots \sqcup \Gamma_{i,n})K \subseteq F, \quad i = 1, 2, ..., n, \tag{7.19}$$

and

(4) if $i \neq j$, then
$$(\Gamma_{i,1} \sqcup \cdots \sqcup \Gamma_{i,n})K \cap (\Gamma_{j,1} \sqcup \cdots \sqcup \Gamma_{j,n}) = \varnothing. \tag{7.20}$$

Set $e = \varphi_B$, and set

$$e_\gamma = u_\gamma^* e u_\gamma, \quad \gamma \in F.$$

For each $1 \leq i \leq n^2$, write $i = n(k-1) + j$, where $1 \leq j \leq n$, and set

$$e_i = \sum_{\gamma \in \Gamma_{k,j}} e_\gamma$$

By (7.18), it follows from Lemma 2.18 that there is a order zero map

$$\phi : \mathrm{M}_{n^2}(\mathbb{C}) \to A$$

such that

$$\phi(e_{i,i}) = e_i, \quad i = 1, 2, ..., n^2.$$

Denote by $E_i := e_i^{-1}((0,1])$ and note that, with $i = n(k-1) + j$ with $1 \leq j \leq n$, one has

$$E_i = \bigsqcup_{\gamma \in \Gamma_{k,j}} B\gamma,$$

and it is clear that

$$\bigsqcup_{i=1}^{n^2} E_i \subseteq \bigsqcup_{\gamma\in F} B\gamma, \tag{7.21}$$

and, by (7.15) and (7.16), one has

$$\mu(\bigsqcup_{\gamma\in F} B\gamma \setminus \bigsqcup_{k=1}^{n^2} E_k) = \mu(\bigsqcup_{\gamma\in R_1\cup R_2} B\gamma) < \lambda\frac{\delta}{16n}\mu(\bigsqcup_{\gamma\in F} B\gamma), \quad \mu\in\mathcal{M}_1(X,\Gamma).$$

This proves Property 1.

Consider

$$b_k := e_{n(k-1)+1} + \cdots + e_{nk}, \quad k = 1, ..., n.$$

Note that, with

$$\Gamma_k := \Gamma_{k,1} \sqcup \cdots \sqcup \Gamma_{k,n},$$

one has

$$b_k = \sum_{\gamma\in\Gamma_k} e_\gamma,$$

and hence, if $\Gamma_k\gamma \subseteq F$ for a group element γ, then

$$u_\gamma^* b_k u_\gamma = \sum_{\gamma'\in\Gamma_k} u_\gamma^* e_{\gamma'} u_\gamma = \sum_{\gamma'\in\Gamma_k} e_{\gamma'\gamma} = \sum_{\gamma'\in\Gamma_k\gamma} e_{\gamma'}$$

Thus, by (7.19), (7.20) and the assumption that $K = K^{-1}$, one has that for any $k_1 \neq k_2$,

$$b_{k_1} \perp u_\gamma b_{k_2} u_\gamma^*, \quad \gamma\in K.$$

This proves Property 3.

Also consider

$$C := \mathrm{C}^*\{u_\gamma f : \gamma\in F, f\in \mathrm{C}_0(B)\} \subseteq A,$$

the C*-algebra of the tower (B, F). Note that, by Lemma 3.12 of [17],

$$C \cong \mathrm{M}_{|F|}(\mathrm{C}_0(B)),$$

and under this isomorphism, for any $g\in \mathrm{C}_0(\bigsqcup_{\gamma\in F} B\gamma) \subseteq \mathrm{C}(X)$, one has $g\in C$ and

$$g \mapsto (x \mapsto \sum_{\gamma\in F} g(x\gamma)e_{\gamma,\gamma}).$$

In particular, since for each $i = 0, 1$, $k = 1, 2, ..., n$ and $m = 1, ..., M$, $b_k\varphi_{O_{i,m}} \in \mathrm{C}_0(\bigsqcup_{\gamma\in F} B\gamma)$, one has

$$b_k\varphi_{O_{i,m}} \in C.$$

Noting that $\Gamma_{i,j}$ are $(\mathcal{F}', \varepsilon')$-invariant, by (7.13) and (7.17), regarding $b_k\varphi_{O_{i,m}}$ as an element of $C \cong \mathrm{M}_{|F|}(\mathrm{C}_0(B))$, one has that for any $x \in B$,

$$\begin{aligned} \operatorname{rank}(b_k\varphi_{O_{i,m}}(x)) &= \left|\left\{\gamma \in F : (b_k\varphi_{O_{i,m}})(x\gamma) > 0\right\}\right| \\ &= \left|\{\gamma \in \Gamma_{k,1} \sqcup \Gamma_{k,2} \sqcup \cdots \sqcup \Gamma_{k,n} : x\gamma \in O_{i,m}\}\right| \\ &\geq \frac{\delta}{2}(|\Gamma_{k,1}| + \cdots + |\Gamma_{k,n}|) \\ &= \frac{\delta}{2} n \left|\Gamma_{k,1}\right| > \frac{\delta}{4n}\left|F\right|; \end{aligned}$$

then, for any $\mu \in \mathcal{M}_1(X, \Gamma)$, by (7.21) in the last step,

$$\begin{aligned} \mu(O_{i,m} \cap \bigsqcup_{j=n(k-1)+1}^{nk} E_j) &= \mu(\left\{x \in X : (b_k\varphi_{O_{i,m}})(x) > 0\right\}) \\ &= \int_B \operatorname{rank}(b_k\varphi_{O_{i,m}}(x))\mathrm{d}\mu \\ &\geq \int_B \frac{\delta}{4n}\left|F\right|\mathrm{d}\mu \\ &= \frac{\delta}{4n}\left|F\right|\mu(B) > \frac{\delta}{4n}\mu(\bigsqcup_{j=1}^{n^2} E_j). \end{aligned} \tag{7.22}$$

Now, for each $i = 0, 1,\ k = 1, 2, ..., n$, let us construct open sets $O^k_{i,m},\ m = 1, 2, ..., M$. Note that (recall $\Gamma_k = \Gamma_{k,1} \sqcup \cdots \sqcup \Gamma_{k,n}$)

$$O_{i,m} \cap \bigcup_{j=n(k-1)+1}^{nk} E_j = O_{i,m} \cap \bigsqcup_{\gamma\in\Gamma_k} B\gamma, \quad i = 0, 1,\ m = 1, 2, ..., M.$$

Consider

$$\Gamma_k^\circ := \Gamma_k \setminus (\Gamma_{k,1} \cup \Gamma_{k,2} \cup \Gamma_{k,3}) = \Gamma_{k,4} \sqcup \Gamma_{k,5} \sqcup \cdots \sqcup \Gamma_{k,n},$$

and define

$$O^k_{0,1} := O_{0,1} \cap \bigsqcup_{\gamma\in \mathrm{int}_{K_0^M}(\Gamma_k^\circ)} B\gamma \quad \text{and} \quad O^k_{0,m} := O^k_{0,1}\kappa_{0,m}, \quad m = 1, 2, ..., M,$$

and

$$O^k_{1,1} := O_{1,1} \cap \bigsqcup_{\gamma\in \mathrm{int}_{K_0^M}(\Gamma_k)} B\gamma \quad \text{and} \quad O^k_{1,m} := O^k_{1,1}\kappa_{1,m}, \quad m = 1, 2, ..., M.$$

Then it is clear that

$$O^k_{0,m} \subseteq O_{0,m} \cap \bigcup_{i=n(k-1)+4}^{nk} E_i \quad \text{and} \quad O^k_{1,m} \subseteq O_{1,m} \cap \bigcup_{j=n(k-1)+1}^{nk} E_j, \quad m = 1, 2, ..., M,$$

Since $\Gamma_{i,j}$ are $(\mathcal{F}', \varepsilon')$-invariant, the sets Γ_k° is also $(\mathcal{F}', \varepsilon')$-invariant. By (7.14), one has

$$\frac{\left|\partial_{K_0^M}(\Gamma_k^\circ)\right|}{|\Gamma_k^\circ|} < \frac{\delta}{16},$$

and therefore, together with (7.12) and (7.22), for any $\mu \in \mathcal{M}_1(X,\Gamma)$ and $i = 0, 1$,

$$\begin{aligned}
\mu(O_{i,1}^k) &\geq \mu(O_{i,1} \cap \bigsqcup_{\gamma \in \mathrm{int}_{K_0^M}(\Gamma_k^\circ)} B\gamma) \\
&\geq \mu(O_{i,1} \cap \bigsqcup_{\gamma \in \Gamma_k^\circ} B\gamma) - \mu(\bigsqcup_{\gamma \in \partial_{K_0^M}(\Gamma_k^\circ)} B\gamma) \\
&\geq \mu(O_{i,1} \cap \bigsqcup_{\gamma \in \Gamma_k^\circ} B\gamma) - \frac{\delta}{16} |\Gamma_k^\circ| \, \mu(B) \\
&> \mu(O_{i,1} \cap \bigsqcup_{j=n(k-1)+4}^{nk} E_j) - \frac{\delta}{16n} \mu(\bigsqcup_{j=1}^{n^2} E_j) \\
&\geq \mu(O_{i,1} \cap \bigsqcup_{j=n(k-1)+1}^{nk} E_j) - \frac{3}{n^2} \mu(\bigsqcup_{j=1}^{n^2} E_j) - \frac{\delta}{16n} \mu(\bigsqcup_{j=1}^{n^2} E_j) \\
&\geq \frac{\delta}{4n} \mu(\bigsqcup_{j=1}^{n^2} E_j) - \frac{\delta}{8n} \mu(\bigsqcup_{j=1}^{n^2} E_j) \\
&= \frac{\delta}{8n} \mu(\bigsqcup_{j=1}^{n^2} E_j).
\end{aligned}$$

This proves Property 2, as desired. ∎

Lemma 7.4. *Let Γ be an infinite discrete amenable group, and let (X,Γ) be a minimal topological dynamical system with the (URP). Let $\lambda > 0$ be arbitrary, and let*

$$O_{0,1}, ..., O_{0,M}, O_{1,1}, ..., O_{1,M} \subseteq X$$

be mutually disjoint non-empty open sets together with

$$\{\kappa_{0,1}(=e), \kappa_{0,2}, ..., \kappa_{0,M}, \kappa_{1,1}(=e), \kappa_{1,2}, ..., \kappa_{1,M}\} \subseteq \Gamma$$

such that

$$O_{i,m} = O_{i,1}\kappa_{i,m}, \quad i = 0, 1, \ m = 1, ..., M.$$

Let $K \subseteq \Gamma$ be a symmetric finite set.

Then there exist $n \in \mathbb{N}$ ($n > 3$) and an order zero c.p.c. map

$$\phi : \mathrm{M}_{n^2}(\mathbb{C}) \to A,$$

where $A = \mathrm{C}(X) \rtimes \Gamma$, such that if

$$h := \phi(1) \quad \text{and} \quad e_i := \phi(e_{i,i}), \quad i = 1, 2, ..., n^2,$$

and

$$b_k := e_{n(k-1)+1} + \cdots + e_{n(k-1)+n}, \quad k = 1, 2, ..., n,$$

then

$$e_i \in \mathrm{C}(X)$$

and if denote by

$$E_i = e^{-1}((0,1]), \quad i = 1, 2, ..., n^2,$$

one has

(1) *for each $k = 1, 2, ..., n$, there are mutually disjoint open sets $O^k_{0,1}, ..., O^k_{0,M}$ and $O^k_{1,1}, ..., O^k_{1,M}$ such that*
   (a) $O^k_{0,m} \subseteq O_{0,m} \cap \bigsqcup_{i=4}^n E_{n(k-1)+i}$ *and* $O^k_{1,m} \subseteq O_{1,m} \cap \bigsqcup_{i=1}^n E_{n(k-1)+i}$, $m = 1, 2, ..., M$,
   (b) $O^k_{i,m} = O^k_{i,1}\kappa_{i,m}$, $i = 0, 1$, $m = 1, 2, ..., M$,
   (c)
$$\lambda\mu(O^k_{0,1}) > \frac{3}{n^2} \quad \text{and} \quad \lambda\mu(O^k_{1,1}) > \mu(X \setminus \bigsqcup_{i=1}^{n^2} E_i), \quad \mu \in \mathcal{M}_1(X, \Gamma),$$

(2)
$$b_{k_1} \perp u_\gamma b_{k_2} u^*_\gamma, \quad \gamma \in K,\ k_1 \neq k_2,\ 1 \leq k_1, k_2 \leq n,$$
where $u_\gamma \in A$ is the canonical unitary of γ.

Proof. Applying Lemma 7.3 with respect to $O_{0,1}, O_{0,2}, ..., O_{0,M}$ and $O_{1,1}, O_{1,2}, ..., O_{1,M}$, and

$$\delta := \min\{\mu(O_{i,m}) : i = 0, 1, m = 1, ..., M,\ \mu \in \mathcal{M}_1(X, \Gamma)\} > 0,$$

one obtains $(\mathcal{F}', \varepsilon')$ and n. Since n can be chosen arbitrarily large, one may assume that

$$\frac{3}{n^2} < \lambda\frac{3\delta}{32n} < \frac{1}{4}. \tag{7.23}$$

Since (X, Γ) is assumed to have the (URP), there exist open towers

$$(B_1, F_1), ..., (B_S, F_S)$$

such that each F_s, $s = 1, ..., S$, is $(\mathcal{F}', \varepsilon')$-invariant and

$$\mu(X \setminus \bigsqcup_{s=1}^{S} \bigsqcup_{\gamma \in F_s} B_s\gamma) < \lambda\frac{\delta}{32n}, \quad \mu \in \mathcal{M}_1(X, \Gamma). \tag{7.24}$$

For each tower (B_s, F_s), since F_s is $(\mathcal{F}', \varepsilon')$-invariant, by Lemma 7.3, there is an order zero c.p.c. map

$$\phi_s : \mathrm{M}_{n^2}(\mathbb{C}) \to A,$$

where $A = \mathrm{C}(X) \rtimes \Gamma$, such that if

$$h_s := \phi_s(1) \quad \text{and} \quad e^{(s)}_i := \phi_s(e_{i,i}), \quad i = 1, 2, ..., n^2,$$

and

$$b_{s,k} := e^{(s)}_{n(k-1)+1} + \cdots + e^{(s)}_{n(k-1)+n}, \quad k = 1, 2, ..., n,$$

then

$$e^{(s)}_i \in \mathrm{C}(X)$$

and if denote by

$$E_{s,i} = (e^{(s)}_i)^{-1}((0,1]), \quad i = 1, 2, ..., n^2,$$

then

(1)
$$\bigsqcup_{i=1}^{n^2} E_{s,i} \subseteq \bigsqcup_{\gamma\in F_s} B_s\gamma,$$
and
$$\mu(\bigsqcup_{\gamma\in F_s} B_s\gamma \setminus \bigsqcup_{i=1}^{n^2} E_{s,i}) < \lambda\frac{\delta}{16n}\mu(\bigsqcup_{\gamma\in F_s} B_s\gamma), \quad \mu\in\mathcal{M}_1(X,\Gamma). \tag{7.25}$$

(2) for each $k=1,2,...,n$, there are open sets $O_{0,1}^{k,s},...,O_{0,M}^{k,s}$ and $O_{1,1}^{k,s},...,O_{1,M}^{k,s}$ such that
  (a) $O_{0,m}^{k,s}\subseteq O_{0,m}\cap\bigsqcup_{j=n(k-1)+4}^{nk}E_{s,j}$ and $O_{1,m}^{k,s}\subseteq O_{1,m}\cap\bigsqcup_{j=n(k-1)+1}^{nk}E_{s,j}$, $m=1,2,...,M$,
  (b) $O_{i,m}^{k,s}=O_{i,1}^{k,s}\kappa_{i,m}$, $i=0,1$, $m=1,2,...,M$,
  (c)
$$\mu(O_{0,1}^{k,s}),\ \mu(O_{1,1}^{k,s}) > \frac{\delta}{8n}\mu(\bigsqcup_{i=1}^{n^2} E_{s,i}), \quad \mu\in\mathcal{M}_1(X,\Gamma). \tag{7.26}$$

(3) $b_{s,k_1}\perp u_\gamma b_{s,k_2}u_\gamma^*$, $\gamma\in K$, $k_1\neq k_2$, $1\leq k_1,k_2\leq n$.

Then, the order zero c.p.c. map
$$\phi := \sum_{s=1}^{S}\phi_s \tag{7.27}$$
is the desired map.

Indeed, it follows (7.27) that
$$h=\phi(1)=\sum_{s=1}^{S}\phi_s(1)=h_1+\cdots+h_S,$$
$$e_i=\phi(e_{i,i})=\sum_{s=1}^{S}\phi_s(e_{i,i})=e_i^{(1)}+\cdots+e_i^{(S)}, \quad i=1,2,...,n^2.$$
In particular,
$$b_k=b_{1,i}+\cdots+b_{S,i}, \quad i=1,2,...,n^2$$
and
$$E_i=e_i^{-1}((0,1])=E_{1,i}\sqcup\cdots\sqcup E_{S,i}, \quad i=1,2,...,n^2.$$
For each $i=0,1$, $k=1,2,...,n$, and $m=1,2,...,M$, set
$$O_{i,m}^k=\bigsqcup_{s=1}^{S}O_{i,m}^{k,s}.$$
By Conditions 2a and 2b, it is clear that
$$O_{0,m}^k\subseteq O_{0,m}\cap\bigsqcup_{j=4}^{n}E_{n(k-1)+j} \quad\text{and}\quad O_{i,m}^k\subseteq O_{i,m}\cap\bigsqcup_{j=1}^{n}E_{n(k-1)+j}$$
and
$$O_{i,m}^k=O_{i,1}^k\kappa_m, \quad i=0,1,\ m=1,2,...,M.$$

By (7.24), (7.25) and (7.23), for any $\mu \in \mathcal{M}_1(X, \Gamma)$,

$$\begin{aligned} \mu(X \setminus \bigsqcup_{i=1}^{n^2} E_i) &= \mu(X \setminus \bigsqcup_{s=1}^{S} \bigsqcup_{i=1}^{n^2} E_{s,i}) \\ &= \mu(X \setminus \bigsqcup_{s=1}^{S} \bigsqcup_{\gamma \in F_s} B_s\gamma) + \sum_{s=1}^{S} \mu(\bigsqcup_{\gamma \in F_s} B_s\gamma \setminus \bigsqcup_{i=1}^{n^2} E_{s,i}) \\ &< \lambda\frac{\delta}{32n} + \sum_{s=1}^{S} \lambda\frac{\delta}{16n}\mu(\bigsqcup_{\gamma \in F_s} B_s\gamma) \\ &< \lambda\frac{\delta}{32n} + \lambda\frac{\delta}{16n} < \lambda\frac{3\delta}{32n} < \frac{1}{4}. \end{aligned} \tag{7.28}$$

and, then by (7.26) and (7.28)

$$\begin{aligned} \lambda\mu(O_{i,1}^k) &= \lambda\sum_{s=1}^{S} \mu(O_{i,1}^{k,s}) > \lambda\frac{\delta}{8n}\sum_{s=1}^{S} \mu(\bigsqcup_{i=1}^{n^2} E_{s,i}) \\ &> \lambda\frac{\delta}{8n}(1 - \frac{1}{4}) = \lambda\frac{3\delta}{32n} \\ &> \mu(X \setminus \bigsqcup_{i=1}^{n^2} E_i). \end{aligned}$$

Also note that, by (7.23),

$$\lambda\mu(O_{0,1}^k) > \lambda\frac{3\delta}{32n} > \frac{3}{n^2}.$$

This verifies Property 1.

Property 2 follows from Condition 3 straightforwardly. This proves the lemma. $\square$

Next, let us perturb further the order zero map ϕ obtained by Lemma 7.4. First, we have the following simple observation.

Lemma 7.5. *Let X be compact Hausdorff space, and let T be a compact set of probability Borel measures.*

(1) *If $O \subseteq X$ is an open set and $\lambda, \delta > 0$ satisfy*

$$\lambda\mu(O) > \delta, \quad \mu \in T,$$

then there is a closed set $D \subseteq O$ such that

$$\lambda\mu(D) > \delta, \quad \mu \in T.$$

(2) *If $O \subseteq X$ is an open set and $C \subseteq X$ is closed set satisfying*

$$\lambda\mu(O) > \mu(C), \quad \mu \in T,$$

for some $\lambda > 0$, then there exist a closed set $D \subseteq O$ and an open set $F \supseteq C$ such that

$$\lambda\mu(D) > \mu(F), \quad \mu \in T.$$

Proof. Let us prove the second statement only. The first statement can be shown with a similar argument.

For any $\mu \in T$, pick continuous functions $f_\mu, g_\mu : X \to [0,1]$ such that $f_\mu|_{X\setminus O} = 0$, $g_\mu|_C = 1$, and

$$\lambda\tau_\mu(f_\mu) > \tau_\mu(g_\mu) + \delta_\mu$$

for some $\delta_\mu > 0$, where $\tau_\mu(f) := \int f d\mu$. Then, pick a open neighborhood N_μ of μ such that

$$\lambda\left|\tau_\mu(f_\mu) - \tau_{\mu'}(f_\mu)\right| < \frac{\delta_\mu}{4} \quad \text{and} \quad \left|\tau_\mu(g_\mu) - \tau_{\mu'}(g_\mu)\right| < \frac{\delta_\mu}{4}, \quad \tau' \in N_\mu,$$

and a straightforward calculation shows

$$\lambda\tau_{\mu'}(f_\mu) > \tau_{\mu'}(g_\mu) + \frac{\delta_\mu}{2}, \quad \mu' \in N_\mu.$$

Since T is compact, there is a finite open cover of T consists of $N_{\mu_1}, ..., N_{\mu_n}$, where $\mu_1, ..., \mu_n \in T$. With

$$f := \max\{f_{\mu_1}, ..., f_{\mu_n}\}, \quad g := \min\{g_{\mu_1}, ..., g_{\mu_n}\}, \quad \text{and} \quad \delta := \frac{1}{2}\min\{\delta_{\mu_1}, ..., \delta_{\mu_n}\},$$

one has

$$f|_{X\setminus O} = 0, \quad g|_C = 1, \quad \text{and} \quad \lambda\tau_\mu(f) > \tau_\mu(g) + \delta, \quad \mu \in T.$$

Then, with a sufficiently small $\varepsilon > 0$, the closed set $D := f^{-1}([\varepsilon, 1])$ and the open set $F := g^{-1}((1-\varepsilon, 1])$ satisfy the lemma. $\square$

Lemma 7.6. *Let Γ be an infinite group, and let (X, Γ) be a minimal topological dynamical system with the (URP). Let $\lambda > 0$ be arbitrary, and let $O_{0,1}, ..., O_{0,M}, O_{1,1}, ..., O_{1,M} \subseteq X$ be mutually disjoint non-empty open sets together with*

$$\{\kappa_{0,1}(=e), \kappa_{0,2}, ..., \kappa_{0,M}, \kappa_{1,1}(=e), \kappa_{1,2}, ..., \kappa_{1,M}\} \subseteq \Gamma$$

such that

$$O_{i,m} = O_{i,1}\kappa_{i,m}, \quad i = 0, 1, \ m = 1, ..., M.$$

Let $K \subseteq \Gamma$ be a symmetric finite set.

Then there is an order zero c.p.c. map

$$\phi : \mathrm{M}_{n^2}(\mathbb{C}) \to A$$

for some $n > 3$ such that if

$$h := \phi(1) \quad \text{and} \quad e_i := \phi(e_{i,i}), \quad i = 1, 2, ..., n^2,$$

and

$$b_k := e_{n(k-1)+1} + \cdots + e_{n(k-1)+n}, \quad k = 1, 2, ..., n,$$

then

$$e_i \in \mathrm{C}(X)$$

and

(1) *for each $k = 1, 2, ..., n$, there are mutually orthogonal positive functions*

$$c_{k,1}, ..., c_{k,M}, d_{k,1}, ..., d_{k,M} \in \mathrm{C}(X)$$

such that

(a) $c_{k,m} \in \mathrm{Her}(O_{0,m})$ *and* $d_{k,m} \in \mathrm{Her}(O_{1,m})$, $m = 1, 2, ..., M$,
(b) $c_{k,m} \perp (e_{(k-1)n+1} + e_{(k-1)n+2} + e_{(k-1)n+3})$, $m = 1, 2, ..., M$,
(c) $c_{k,m} b_k = c_{k,m}$ *and* $d_{k,m} b_k = d_{k,m}$, $m = 1, 2, ..., M$,
(d) $c_{k,m} = u^*_{\kappa_m} c_{k,1} u_{\kappa_m}$ *and* $d_{k,m} = u^*_{\kappa_m} d_{k,1} u_{\kappa_m}$, $m = 1, 2, ..., M$, *and*
(e) $\lambda \mathrm{d}_\tau(c_{k,1}) > \frac{3}{n^2}$ *and* $\lambda \mathrm{d}_\tau(d_{k,1}) > \mathrm{d}_\tau(1-h)$, $\tau \in \mathrm{T}(A)$,

(2)
$$b_{k_1} \perp u_\gamma b_{k_2} u^*_\gamma, \quad \gamma \in K,\ k_1 \neq k_2,\ 1 \leq k_1, k_2 \leq n,$$
where $u_\gamma \in A$ *is the canonical unitary of* γ.

Proof. It follows form Lemma 7.4 that there exist $n \in \mathbb{N}$ $(n > 3)$ and an order zero c.p.c. map
$$\phi' : \mathrm{M}_{n^2}(\mathbb{C}) \to A$$
such that if
$$h' := \phi'(1) \quad \text{and} \quad e'_i := \phi'(e_{i,i}), \quad i = 1, 2, ..., n^2,$$
and
$$b'_k := e'_{n(k-1)+1} + \cdots + e'_{n(k-1)+n}, \quad k = 1, 2, ..., n,$$
then
$$e'_i \in \mathrm{C}(X)$$
and if denote by
$$E_i = (e'_i)^{-1}((0, 1]), \quad i = 1, 2, ..., n^2,$$
then

(1) for each $k = 1, 2, ..., n$, there are mutually disjoint open sets $O^k_{0,1}, ..., O^k_{0,M}$ and $O^k_{1,1}, ..., O^k_{1,M}$ such that
(a) $O^k_{0,m} \subseteq O_{0,m} \cap \bigsqcup_{i=4}^n E_{n(k-1)+i}$ and $O^k_{1,m} \subseteq O_{1,m} \cap \bigsqcup_{i=1}^n E_{n(k-1)+i}$, $m = 1, 2, ..., M$,
(b) $O^k_{i,m} = O^k_{i,1} \kappa_{i,m}$, $i = 0, 1$, $m = 1, 2, ..., M$,
(c)
$$\lambda \mu(O^k_{0,1}) > \frac{3}{n^2} \quad \text{and} \quad \lambda \mu(O^k_{1,1}) > \mu(X \setminus \bigsqcup_{i=1}^{n^2} E_i), \quad \mu \in \mathcal{M}_1(X, \Gamma),$$

(2)
$$b'_{k_1} \perp u_\gamma b'_{k_2} u^*_\gamma, \quad \gamma \in K,\ k_1 \neq k_2,\ 1 \leq k_1, k_2 \leq n.$$

Since $\mathcal{M}_1(X, \Gamma)$ is compact, $O^k_{0,1}$ and $O^k_{1,1}$ are open, and $X \setminus \bigsqcup_{i=1}^{n^2} E_i$ is closed, by Condition (1c) and Lemma 7.5, there are closed sets $D^k_{i,1} \subseteq O^k_{i,1}$, $i = 0, 1$, and an open set $U \supseteq X \setminus \bigsqcup_{i=1}^{n^2} E_i$ such that
$$\lambda \mu(D^k_{0,1}) > \frac{3}{n^2} \quad \text{and} \quad \lambda \mu(D^k_{1,1}) > \mu(U), \quad \mu \in \mathcal{M}_1(X, \Gamma). \tag{7.29}$$

For any $\varepsilon > 0$, define
$$V_\varepsilon := \mathrm{int}(f_\varepsilon(h')^{-1}(\{1\})) = \{x \in X : h'(x) > \varepsilon\}, \tag{7.30}$$
and consider the open sets
$$W^k_{i,\varepsilon} := (O^k_{i,1} \cap V_\varepsilon) \cap (O^k_{i,2} \cap V_\varepsilon)\kappa^{-1}_{i,2} \cap \cdots \cap (O^k_{i,M} \cap V_\varepsilon)\kappa^{-1}_{i,M}, \tag{7.31}$$

which increases to $O^k_{i,1}$ as $\varepsilon \to 0$, $i = 0, 1$. Since $D^k_{i,1}$ is compact, there is a sufficiently small $\varepsilon > 0$ such that

$$W^k_{i,\varepsilon} \supseteq D^k_{i,1}, \quad i = 0, 1.$$

Pick a such ε, and one may also assume that

$$U \supseteq \{x \in X : h'(x) < \varepsilon\},$$

and then note

$$U \supseteq \{x \in X : h'(x) < \varepsilon\} = (1 - f_\varepsilon(h'))^{-1}((0, 1]).$$

Then, together with (7.29),

$$\lambda\mu(W^k_{0,\varepsilon}) > \lambda\mu(D^k_{0,1}) > \frac{3}{n^2}, \quad \mu \in \mathcal{M}_1(X, \Gamma), \tag{7.32}$$

and

$$\lambda\mu(W^k_{1,\varepsilon}) > \lambda\mu(D^k_{1,1}) > \mu(U) \geq \mu((1 - f_\varepsilon(h'))^{-1}((0, 1])), \quad \mu \in \mathcal{M}_1(X, \Gamma), \tag{7.33}$$

It also follows from (7.31) that

$$W^k_{i,\varepsilon}\kappa_{i,m} \subseteq V_\varepsilon, \quad i = 0, 1,\ m = 1, 2, ..., M. \tag{7.34}$$

Set

$$c_{k,1} = \varphi_{W^k_{0,\varepsilon}} \quad \text{and} \quad c_{k,m} = u^*_{\kappa_m} c_{k,1} u_{\kappa_m}, \quad m = 2, 3, ..., M,$$

and

$$d_{k,1} = \varphi_{W^k_{1,\varepsilon}} \quad \text{and} \quad c_{k,m} = u^*_{\kappa_m} c_{k,1} u_{\kappa_m}, \quad m = 2, 3, ..., M.$$

Note that, by (7.31),

$$c_{k,m} \in \mathrm{Her}(O^k_{0,m}) \quad \text{and} \quad d_{k,m} \in \mathrm{Her}(O^k_{1,m}), \quad m = 1, 2, ..., M.$$

It follows from (7.34) and (7.30) that

$$c_{k,m} f_\varepsilon(b'_k) = c_{k,m} f_\varepsilon(h') = c_{k,m} \quad \text{and} \quad d_{k,m} f_\varepsilon(b'_k) = d_{k,m} f_\varepsilon(h') = d_{k,m},$$

and it follows from (7.32) and (7.33) that for any $\tau \in \mathrm{T}(A)$,

$$\lambda \mathrm{d}_\tau(c_{k,1}) = \lambda\mu_\tau(W^k_{0,\varepsilon}) > \frac{3}{n^2}.$$

and

$$\lambda \mathrm{d}_\tau(d_{k,1}) \;=\; \lambda\mu_\tau(W^k_{1,\varepsilon}) > \mu_\tau((1 - f_\varepsilon(h'))^{-1}((0, 1])) = \mathrm{d}_\tau(1 - f_\varepsilon(h')).$$

Then

$$\phi := f_\varepsilon(\phi') : \mathrm{M}_{n^2}(\mathbb{C}) \to A$$

is the desired order-zero map.

Indeed, noting that

$$h = \phi(1) = f_\varepsilon(h'),$$

the existence of $c_{k,m}$, $d_{k,m}$, $k = 1, ..., n$, $m = 1, ..., M$, and Property 1 are verified above.

Consider any b_{k_1}, b_{k_2} with $k_1 \neq k_2$, $1 \leq k_1, k_2 \leq n$. Note that

$$b_{k_1} = f_\varepsilon(b'_{k_1}) \in \mathrm{Her}(b'_{k_1}) \quad \text{and} \quad u_\gamma b_{k_2} u^*_\gamma = f_\varepsilon(u_\gamma b'_{k_2} u^*_\gamma) \in \mathrm{Her}(u_\gamma b'_{k_2} u^*_\gamma), \quad \gamma \in K,$$

and therefore, it follows from Condition 2 that

$$b_{k_1} \perp u_\gamma b_{k_2} u_\gamma^*, \quad \gamma \in K.$$

This verified Property 2, as desired. □

We are now ready for the main results of the paper.

Proposition 7.7. *Let Γ be an infinite countable discrete amenable group, and let (X, Γ) be a minimal free topological dynamical system with the (URP) and (COS). Then the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ has Property (D).*

Proof. Let $a \in \mathrm{ZD}(A)$ and let $\varepsilon > 0$ be arbitrary. It follows from Proposition 6.2 that there are unitaries $u_1, u_2 \in A$, $a' \in \mathrm{C_c}(\Gamma, \mathrm{C}(X))$ and a non-empty open set $E \subseteq X$ such that

$$\|u_1 a u_2 - a'\| < \varepsilon \quad \text{and} \quad \varphi_E a' = a' \varphi_E = 0.$$

In the following, let us verify that a' is actually a $\mathcal{D}_0$-operator. Since ε is arbitary, this shows that A has Property (D).

Note that, since (X, Γ) has the (COS), the sub-C*-algebra $\mathrm{C}(X)$ has the (λ, M)-Cuntz comparison insider A for some $\lambda \in (0, +\infty)$ and $M \in \mathbb{N}$. Fix λ and M.

Write

$$a' = \sum_{\gamma \in K} f_\gamma u_\gamma, \tag{7.35}$$

where $f_\gamma \in \mathrm{C}(X)$ and $K \subseteq \Gamma$ is a symmetric finite set.

Consider the open set E. Since (X, Γ) is minimal, all orbits are dense, and hence there exist non-empty mutually orthogonal open sets

$$O_{0,1}, ..., O_{0,M}, O_{1,1}, ..., O_{1,M} \subseteq E$$

and

$$\{\kappa_{0,1}(= e), \kappa_{0,2}, ..., \kappa_{0,M}, \kappa_{1,1}(= e), \kappa_{1,2}, ..., \kappa_{1,M}\} \subseteq \Gamma \tag{7.36}$$

such that

$$O_{i,m} = O_{i,1} \kappa_{i,m}, \quad i = 0, 1, \ m = 1, ..., M.$$

Since (X, Γ) has the (URP), it follows from Lemma 7.6 that there is an order zero c.p.c. map

$$\phi : \mathrm{M}_{n^2}(\mathbb{C}) \to A$$

for some $n > 3$ such that if

$$h := \phi(1), \quad e_i := \phi(e_{i,i}), \quad i = 1, 2, ..., n^2,$$

$$s_k := e_{(k-1)p+1} + \cdots + e_{(k-1)p+3}, \quad k = 1, ..., n,$$

and

$$E_k := e_{n(k-1)+1} + \cdots + e_{n(k-1)+n}, \quad k = 1, 2, ..., n,$$

then

$$e_i \in \mathrm{C}(X) \tag{7.37}$$

and

(1) for each $k = 1, 2, ..., n$, there are mutually orthogonal positive functions

$$c_{k,1}, ..., c_{k,M}, d_{k,1}, ..., d_{k,M} \in \mathrm{C}(X)$$

such that

(a) $c_{k,m} \in \mathrm{Her}(O_{0,m})$ and $d_{k,m} \in \mathrm{Her}(O_{1,m})$, $m = 1, 2, ..., M$,
(b) $c_{k,m} \perp s_k$, $m = 1, 2, ..., M$,
(c) $c_{k,m} E_k = c_{k,m}$ and $d_{k,m} E_k = d_{k,m}$, $m = 1, 2, ..., M$,
(d) $c_{k,m} = u^*_{\kappa_m} c_{k,1} u_{\kappa_m}$ and $d_{k,m} = u^*_{\kappa_m} d_{k,1} u_{\kappa_m}$, $m = 1, 2, ..., M$, and
(e) $\lambda \mathrm{d}_\tau(c_{k,1}) > \frac{3}{n^2}$ and $\lambda \mathrm{d}_\tau(d_{k,1}) > \mathrm{d}_\tau(1 - h)$, $\tau \in \mathrm{T}(A)$,

(2)

$$E_{k_1} \perp u_\gamma E_{k_2} u^*_\gamma, \quad \gamma \in K,\ k_1 \neq k_2,\ 1 \leq k_1, k_2 \leq n,$$

where $u_\gamma \in A$ is the canonical unitary of γ.

Let us verify that the order zero map ϕ satisfies Definition 5.1 with $p = q = n$, $l = 1$, and $r = 3$. (With the given p, q, l, r, it is straightforward to verify that 2 of Definition 5.1 holds.)

Note that, by Equations (7.35), (7.37), and Condition 2, for any $k_1 \neq k_2$, $1 \leq k_1, k_2 \leq n$,

$$\begin{aligned} E_{k_1} a' E_{k_2} &= E_{k_1}(\sum_{\gamma \in K} f_\gamma u_\gamma) E_{k_2} = \sum_{\gamma \in K} E_{k_1} f_\gamma u_\gamma E_{k_2} \\ &= \sum_{\gamma \in K} f_\gamma E_{k_1} u_\gamma E_{k_2} \\ &= \sum_{\gamma \in K} f_\gamma (E_{k_1} u_\gamma E_{k_2} u^*_\gamma) u_\gamma = 0. \end{aligned}$$

In particular, this verifies 1 of Definition 5.1.

Set

$$c_k = c_{k,1} + \cdots + c_{k,M} \quad \text{and} \quad d_k = d_{k,1} + \cdots + d_{k,M}, \quad k = 1, ..., n.$$

Then, 3a 3b 3c of Definition 5.1 follows directly from Conditions 1a, 1b, and 1c above.

As for 3d of Definition 5.1, note that it follows from Condition 1e above that

$$\mathrm{d}_\tau(s_k) \leq \frac{3}{n^2} < \lambda \mathrm{d}_\tau(c_{k,1}) \quad \text{and} \quad \mathrm{d}_\tau(1 - h) < \lambda \mathrm{d}_\tau(d_{k,1}), \quad \tau \in \mathrm{T}(A).$$

Since (X, Γ) has (λ, M)-Cuntz comparison of open sets and $c_{k,1}, d_{k,1}, h, s_k \in \mathrm{C}(X)$, one has

$$s_k \precsim \underbrace{c_{k,1} \oplus \cdots \oplus c_{k,1}}_{M} \quad \text{and} \quad 1 - h \precsim \underbrace{d_{k,1} \oplus \cdots \oplus d_{k,1}}_{M}.$$

By Condition 1d above, the positive elements $c_{k,m}$, $m = 1, ..., M$, are mutually orthogonal and mutually Cuntz equivalent, and the positive elements $d_{k,m}$, $m = 1, ..., M$ are mutually orthogonal and mutually Cuntz equivalent. One then has

$$c_k \sim \underbrace{c_{k,1} \oplus \cdots \oplus c_{k,1}}_{M} \quad \text{and} \quad d_k \sim \underbrace{d_{k,1} \oplus \cdots \oplus d_{k,1}}_{M},$$

and hence

$$s_k \precsim c_k \quad \text{and} \quad 1 - h \precsim d_k.$$

This shows that a' is a $\mathcal{D}_0$-operator, as desired. $\square$

Theorem 7.8. *Let Γ be a countable discrete amenable group, and let (X, Γ) be a free and minimal topological dynamical system with the (URP) and (COS). Then* $\mathrm{tsr}(\mathrm{C}(X) \rtimes \Gamma) = 1$.

Proof. If $|\Gamma| < \infty$, since (X, Γ) is minimal, the space X must consist of finitely many points and $\mathrm{C}(X) \rtimes \Gamma \cong \mathrm{M}_{|\Gamma|}(\mathbb{C})$. In particular, it has stable rank one.

If $|\Gamma| = \infty$, then it follows from Proposition 7.7 that $\mathrm{C}(X) \rtimes \Gamma$ has Property (D). Since $\mathrm{C}(X) \rtimes \Gamma$ is finite, it follows from Theorem 5.5 that $\mathrm{tsr}(\mathrm{C}(X) \rtimes \Gamma) = 1$, as desired. ☐

Corollary 7.9. *Let $(X, \mathbb{Z}^d)$ be a free and minimal topological dynamical system. Then* $\mathrm{tsr}(\mathrm{C}(X) \rtimes \mathbb{Z}^d) = 1$.

Proof. By Theorem 4.2 and Theorem 5.5 of [16], any free and minimal dynamical system $(X, \mathbb{Z}^d)$ has the (URP) and (COS). It then follows from Theorem 7.8 that $\mathrm{tsr}(\mathrm{C}(X) \rtimes \mathbb{Z}^d) = 1$. ☐

Remark 7.10. Without simplicity, the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ might not have stable rank one in general, even if X is the Cantor set, $\Gamma = \mathbb{Z}$, and $(X, \mathbb{Z})$ has finitely many minimal closed invariant subsets (see, [19] or [3]).

Corollary 7.11. *Let $(X, \mathbb{Z}^d)$ be a free and minimal dynamical system, and set $A = \mathrm{C}(X) \rtimes \mathbb{Z}^d$. Then*

(1) *A has cancellation of projections, i.e., if $p, q \in A \otimes \mathcal{K}$ are two projections such that $p \oplus r \sim q \oplus r$ for some projections $r \in A \otimes \mathcal{K}$, then $p \sim q$.*
(2) *A has cancellation in Cuntz semigroup: let $x, y \in \mathrm{W}(A)$ such that $x + [c] \leq y + [(c - \varepsilon)_+]$ for some positive element $c \in \mathrm{M}_\infty(A)$, then $x \leq y$.*
(3) *The canonical map $\mathrm{U}(A)/\mathrm{U}_0(A) \to \mathrm{K}_1(A)$ is an isomorphism. That is, any unitary of $A \otimes \mathcal{K}$ is homotopic to a unitary of A, and if a unitary u of A is connected to the identity with a path of unitaries of $A \otimes K$, then u can be connected to the identity by a path of unitaries of A.*

Proof. Statements 1 and 3 are well known fact for C*-algebras with stable rank one ([21]). Statement 2 follows from Theorem 4.3 of [25] (an earlier version were obtained in [8]). ☐

By Theorem 4.1 of [5], the Cuntz semigroup classifies homomorphisms from an inductive limit of interval algebras (AI algebra) to a C*-algebra A with stable rank one. Therefore we have the following corollary.

Corollary 7.12. *Let $(X, \mathbb{Z}^d)$ be a free and minimal dynamical system. Let $\phi_1, \phi_2 : I \to A = \mathrm{C}(X) \rtimes \mathbb{Z}^d$ be two homomorphisms, where I is an AI algebra. Then ϕ_1 and ϕ_2 are approximately unitarily equivalent if, and only if, $[\phi_1] = [\phi_2]$ on the Cuntz semigroups.*

The next corollary follows from [27]:

Corollary 7.13. *Let (X, Γ) be a free and minimal dynamical system with the (URP) and (COS). Then for every $f \in \mathrm{LAff}(\mathrm{T}(A))_{++}$, where $A = \mathrm{C}(X) \rtimes \Gamma$, there exists $a \in (A \otimes \mathcal{K})^+$ such that*

$$\mathrm{d}_\tau(a) = f(\tau), \quad \tau \in \mathrm{T}(A).$$

Moreover, if A has strict comparison of positive elements, then the Cuntz semigroup of A is almost divisible (see [27]*). In this case, there are canonical order-isomorphisms*

$$\mathrm{Cu}(A) \cong V(A) \sqcup \mathrm{LAff}(\mathrm{T}(A))_{++} \cong \mathrm{Cu}(\mathrm{A} \otimes \mathcal{Z}).$$

In particular, the statements above hold for $\Gamma = \mathbb{Z}^d$.

Proof. This follows directly from Theorem 8.11 and Corollary 8.12 of [27]. □

In fact, if $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ has strict comparison of positive elements, then it actually is Jiang-Su stable:

Corollary 7.14. *Let (X, Γ) be a free and minimal dynamical system with the (URP) and (COS), and denote by $A = \mathrm{C}(X) \rtimes \Gamma$. Then $A \cong A \otimes \mathcal{Z}$ if, and only if, A has strict comparison of positive elements (that is, it satisfies the Toms-Winter conjecture). In particular, the statement holds for $\Gamma = \mathbb{Z}^d$.*

Proof. One only need to show the "if" part. Since A is assumed to have strict comparison of positive elements, by Corollary 7.13, one has that $\mathrm{Cu}(A) \cong V(A) \sqcup \mathrm{LAff}(\mathrm{T}(A))_{++}$ and hence it is tracially 0-divisible (see Corollary 2.6 of [18] and its proof). It then follows from Proposition 3.8 of [18] and the strict comparison assumption that A is tracially $\mathcal{Z}$-stable. Since A is nuclear, one has that $A \cong A \otimes \mathcal{Z}$, as desired. □

Since the real rank of a C*-algebra A is at most $2 \cdot \mathrm{tsr}(A) - 1$, one has the following estimate:

Corollary 7.15. *Let $(X, \mathbb{Z}^d)$ be a free and minimal dynamical system. The real rank of $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ is either 0 or 1.*

Remark 7.16. Consider a simple unital AH algebra A with diagonal maps. It is known that if A has real rank zero (or just projections separate traces), then A is classifiable ([15]). Does the same statement hold for the crossed-product C*-algebras $\mathrm{C}(X) \rtimes \mathbb{Z}$ (or $\mathrm{C}(X) \rtimes \Gamma$ in general)? That is, if $\mathrm{C}(X) \rtimes \mathbb{Z}$ (or $\mathrm{C}(X) \rtimes \Gamma$, in general) has real rank zero, does $\mathrm{C}(X) \rtimes \mathbb{Z}$ (or $\mathrm{C}(X) \rtimes \Gamma$, in general) absorb the Jiang-Su algebra $\mathcal{Z}$ tensorially? What if one only assumes that projections separate traces instead of real rank zero?

Let Γ be a countable discrete group with sub-exponential growth, and let (X, Γ) be a free and minimal dynamical system. Assume that (X, Γ) is an extension of a minimal Γ-action on a Cantor set. Then it was shown in [26] that the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ has stable rank one. Note that, by Corollary 3.8 and Corollary 8.11 of [17], the dynamical system (X, Γ) has the (URP) and (COS), and therefore this result also can follows from Theorem 7.8.

Corollary 7.17 (c.f. Main Theorem of [26])**.** *Let Γ be a countable discrete group with sub-exponential growth, let (X, Γ) be a free and minimal dynamical system. Assume that (X, Γ) is an extension of a Γ-action on the Cantor set. Then $\mathrm{tsr}(\mathrm{C}(X) \rtimes \Gamma) = 1$.*

8. Two remarks on Property (D)

In the final section, let us remark that simple $\mathcal{Z}$-stable C*-algebras and simple AH-algebras with diagonal maps all have Property (D). These C*-algebras (if finite for the case of $\mathcal{Z}$-stable C*-algebras) are known to have stable rank one (see [24] and [9]).

8.1. **$\mathcal{Z}$-stable C*-algebras.** Let A be a unital simple exact C*-algebra such that $A\cong A\otimes\mathcal{Z}$, where $\mathcal{Z}$ is the Jiang-Su algebra. Note that A has strict comparison of positive elements (we include purely infinite C*-algebras, which have empty tracial simplices).

Let $a\in\mathrm{ZD}(A)$ with $\|a\|=1$ and let $\varepsilon>0$ ber arbitrary. Pick $d_1, d_2\in A^+$ such that $\|d_1\|=\|d_2\|=1$ and

$$d_1a=ad_2=0.$$

By regarding A as $A\otimes\mathcal{Z}\otimes\mathcal{Z}\otimes\mathcal{Z}$, one obtains $\tilde{a}, \tilde{d_1}, \tilde{d_2}\in A\otimes\mathcal{Z}\otimes 1\otimes 1$ with norm one and a unitary $u\in A\otimes\mathcal{Z}\otimes\mathcal{Z}\otimes 1$ such that

$$\|uau^*-\tilde{a}\|<\frac{\varepsilon}{12},\quad \left\|ud_1u^*-\tilde{d_1}\right\|<\frac{\varepsilon}{12},\quad \left\|ud_2u^*-\tilde{d_2}\right\|<\frac{\varepsilon}{12},$$

and then

$$\left\|\tilde{d_1}\tilde{a}\right\|<\frac{\varepsilon}{6}\quad\text{and}\quad\left\|\tilde{a}\tilde{d_2}\right\|<\frac{\varepsilon}{6}.$$

With a small perturbation of $\tilde{d_1}$ and $\tilde{d_2}$, one may assume that there are positive elements $d_1', d_2'\in A\otimes\mathcal{Z}\otimes 1\otimes 1$ with $\|d_1'\|=\|d_2'\|=1$ such that

$$d_1'\tilde{d_1}=d_1'\quad\text{and}\quad d_2'\tilde{d_2}=d_2'$$

Note that

$$(1-\tilde{d_1})\tilde{a}(1-\tilde{d_2})\approx_{\frac{\varepsilon}{2}}\tilde{a}\quad\text{and}\quad d_1'(1-\tilde{d_1})\tilde{a}(1-\tilde{d_2})=(1-\tilde{d_1})\tilde{a}(1-\tilde{d_2})d_2'=0.$$

Pick two orthogonal nonzero positive elements $s_1, s_2\in 1\otimes 1\otimes\mathcal{Z}\otimes 1$, and consider the positive elements $\tilde{d_1}s_1$ and $\tilde{d_2}s_2$. Since s_1, s_2 commute with $\tilde{d_1}, \tilde{d_2}$, one has that

$$\tilde{d_1}s_1\perp\tilde{d_2}s_2\quad\text{and}\quad(\tilde{d_1}s_1)((1-d_1')\tilde{a}(1-\tilde{d_2}))=((1-\tilde{d_1})\tilde{a}(1-\tilde{d_2}))(d_2's_2)=0.$$

Since $A\otimes\mathcal{Z}\otimes\mathcal{Z}\otimes 1$ is simple, there is $v\in A\otimes\mathcal{Z}\otimes\mathcal{Z}\otimes 1$ with $\|v\|=1$ such that

$$vv^*\in\mathrm{Her}(d_1s_1)\quad\text{and}\quad v^*v\in\mathrm{Her}(d_2s_2),$$

and moreover, with the polar decomposition and a further perturbation, one may assume that there is a positive element b such that $\|b\|=1$ and $(vv^*)b=b$ (hence $b\in\mathrm{Her}(d_1s_1)$). It then follows from Lemma 3.2 that there is a unitary $w\in A\otimes\mathcal{Z}\otimes\mathcal{Z}\otimes 1$ such that

$$wbw^*\in\mathrm{Her}(d_2s_2).$$

Thus, with

$$a':=(1-\tilde{d_1})\tilde{a}(1-\tilde{d_2})w,$$

one has

$$\|uau^*w-a'\|<\varepsilon\quad\text{and}\quad ba'=0=a'b.$$

Let us show that a' is a $\mathcal{D}_0$-operator, and thus A has Property (D).

Since A is not of type I, there are positive elements $c, d \in \overline{b(A \otimes \mathcal{Z} \otimes \mathcal{Z} \otimes 1)b}$ such that $c \perp d$ and $\|c\| = \|d\| = 1$. Since A is simple, there is $\delta > 0$ such that

$$\tau(c), \tau(d) > \delta, \quad \tau \in \mathrm{T}(A).$$

Now, consider the embedding $\phi' : \mathcal{Z} \to 1 \otimes 1 \otimes 1 \otimes \mathcal{Z}$, and note that

$$[a', \phi'(a)] = 0 \quad \text{and} \quad [b, \phi'(a)] = 0, \quad a \in \mathcal{Z}. \tag{8.1}$$

Pick $n \in \mathbb{N}$ sufficiently large such that

$$(n-3)\delta > 6,$$

and pick a standard embedding

$$\iota : \mathrm{M}_{n^2}(\mathrm{C}_0((0,1])) \to \mathcal{Z}_{n^2, n^2+1} \to \mathcal{Z}.$$

Denote by ϕ'' the order zero map induced by the homomorphism $\phi' \circ \iota$, and choose $\varepsilon' > 0$ sufficiently small so that

$$\tau(f_{\varepsilon'}(\phi''(1))) > 1 - \delta/2n, \quad \tau \in \mathrm{T}(A).$$

For each $k = 1, 2, ..., n$, define

$$c_k = c \cdot f_{\varepsilon'}(\phi'')(e_{(k-1)n+4} + \cdots + e_{(k-1)n+n}),$$

and

$$d_k = d \cdot f_{\varepsilon'}(\phi'')(e_{(k-1)n+1} + \cdots + e_{(k-1)n+n}).$$

A straightforward calculation (using (8.1)) shows that c_k, $d_k \in \overline{b(A \otimes \mathcal{Z} \otimes \mathcal{Z} \otimes \mathcal{Z})b}$, $c_k \perp d_k$ and

$$c_k E_k = c_k \quad \text{and} \quad d_k E_k = d_k,$$

where $E_k := f_{\frac{\varepsilon'}{2}}(\phi'')(e_{(k-1)n+1} + \cdots + e_{(k-1)n+n})$.

Note that, for any $\tau \in \mathrm{T}(A)$,

$$\mathrm{d}_\tau(c_k) > \delta \cdot \frac{n-3}{n^2} \cdot \frac{2n-\delta}{2n} > \frac{3}{n^2} \cdot \frac{4n-2}{2n} > \frac{3}{n^2}$$

and

$$\mathrm{d}_\tau(d_k) > \delta \cdot \frac{1}{n} \cdot \frac{2n-\delta}{2n} > \delta \cdot \frac{1}{n} \cdot \frac{2n-1}{2n} > \frac{\delta}{2n} > \mathrm{d}_\tau(1 - f_{\frac{\varepsilon}{2}}(\phi''(1))).$$

Since A has strict comparison of positive elements, one has that

$$s_k \precsim c_k \quad \text{and} \quad 1 - f_{\frac{\varepsilon}{2}}(\phi''(1)) \precsim d_k.$$

This shows that a' is a $\mathcal{D}_0$-operator (with $\phi = f_{\frac{\varepsilon'}{2}}(\phi'')$, $p = q = n$, $r = 3$, and $l = 1$ in Definition 5.1).

8.2. **AH algebras with diagonal maps.** Recall that an AH algebra with diagonal maps is the limit of a unital inductive sequence (A_n, ψ_n), where

$$A_n = \bigoplus_{i=1}^{h_n} \mathrm{M}_{k_{n,i}}(\mathrm{C}(X_{n,i}))$$

for some compact metrizable space $X_{n,i}$, and if

$$D_n := \bigoplus_i \{\mathrm{diag}\{f_1, f_2, ..., f_{k_{n,i}}\} : f_k \in \mathrm{C}(X_{n,i})\} \subseteq \bigoplus_i \mathrm{M}_{k_{n,i}}(\mathrm{C}(X_{n,i})) = A_n,$$

then

$$\psi_n(D_n) \subseteq D_{n+1}.$$

Let A be simple AH algebra with diagonal maps. It then follows from Theorem 3.4 of [9] that A has Property (D), and we leave the details to readers. Alternatively, let us propose the following approach which is in the similar line of our approach to the crossed product C*-algebra $\mathrm{C}(X) \rtimes \Gamma$: Consider

$$D := \varinjlim D_n \subseteq \varinjlim A_n = A.$$

Then the commutative sub-C*-algebra D actually behaves like the sub-C*-algebra $\mathrm{C}(X)$ of $\mathrm{C}(X) \rtimes \Gamma$.

Let $a \in A$ satisfy $\|a\| = 1$ and $d_1 a = a d_2 = 0$ for some nonzero positive elements d_1, d_2, and let $\varepsilon > 0$ be arbitrary. With a telescoping of the inductive sequence if necessary, there are $\tilde{a}, \tilde{d}_1, \tilde{d}_2 \in A_1$ with norm one such that

$$\|a - \tilde{a}\| < \frac{\varepsilon}{2} \quad \text{and} \quad \left\|\tilde{d}_1 \tilde{a}\right\|, \left\|\tilde{a}\tilde{d}_1\right\| < \frac{\varepsilon}{12},$$

where $\tilde{d}_1, \tilde{d}_2$ are positive. Since $\tilde{d}_1, \tilde{d}_2$ has norm one, there is $x_0 \in X_{1,i}$ for some i such that

$$\left\|\tilde{d}_1(x_0)\right\| = 1 \quad \text{and} \quad \left\|\tilde{d}_2(x_0)\right\| = 1.$$

Since $\tilde{d}_1, \tilde{d}_2$ are positive, by conjugating some constant unitary matrices, one may assume that $\tilde{d}_1$ and $\tilde{d}_2$ are diagonal matrices at x_0. Hence, by cutting the diagonal entry which has value 1 at x_0, one can find a positive element $h \in D_1$ which is constantly 1 on a small neighborhood of x_0 such that

$$h\tilde{d}_1 \approx_{\frac{\varepsilon}{12}} h \quad \text{and} \quad h\tilde{d}_2 \approx_{\frac{\varepsilon}{12}} h.$$

Then a straightforward calculation shows that

$$\|h\tilde{a}\| < \frac{\varepsilon}{6} \quad \text{and} \quad \|\tilde{a}h\| < \frac{\varepsilon}{6}.$$

Since h is constantly 1 on a neighborhood of x_0, there is a positive element $b \in D_n$ with norm 1 such that $bh = b$. Then

$$a' := (1 - h)\tilde{a}(1 - h)$$

and b satisfy

$$\|a - a'\| < \varepsilon \quad \text{and} \quad ba' = a'b = 0.$$

Now, let us show that a' is a $\mathcal{D}_0$-operator, and thus A has Property (D).

Choose positive orthogonal functions $c, d \in D_1$ such that $c, d \in bD_1b$. Set

$$\delta = \min\{\tau(c), \tau(d); \tau \in \mathrm{T}(A)\} > 0.$$

By another telescoping if necessary, one may assume that

$$\frac{3}{k_{1,i}} < \frac{\delta}{2} < \frac{\min\{\mathrm{rank}(c(x)), \mathrm{rank}(d(x))\}}{k_{1,i}}, \quad x \in X_{1,i}. \tag{8.2}$$

Choose $l \in \mathbb{N}$ such that

$$\frac{12}{l-3} < \frac{\delta}{2}.$$

Set $K := \max\{k_{1,1}, ..., k_{1,h_1}\} + 1$. Consider A_2, and to simplify notation, rewrite $A_2 = \bigoplus_{s=1}^{S} \mathrm{M}_{k_s}(\mathrm{C}(X_s))$. With a telescoping of the inductive sequence if necessary, one has that, insider each direct summand of A_2,

(1) the element a' is a matrix of continuous functions with

$$a'_{i,j} = 0, \quad \text{if } |i-j| \geq K, \tag{8.3}$$

(2) write

$$c = \bigoplus_{s=1}^{S} \mathrm{diag}\{c_1^{(s)}, ..., c_{k_s}^{(s)}\} \quad \text{and} \quad d = \bigoplus_{s=1}^{S} \mathrm{diag}\{d_1^{(s)}, ..., d_{k_s}^{(s)}\},$$

where $c_i^{(s)}, d_i^{(s)} \in \mathrm{C}(X_s)$, then, by (8.2), for any $L = 1, ..., k_s$,

$$\frac{\delta}{2}(1 - \frac{2K}{L}) < \frac{\left|\{i_0 \leq i \leq i + L - 1 : c_i^{(s)}(x) \neq 0\}\right|}{L}, \quad 1 \leq i_0 < k_s - L,\ x \in X_s, \tag{8.4}$$

and

$$\frac{\delta}{2}(1 - \frac{2K}{L}) < \frac{\left|\{i_0 \leq i \leq i + L - 1 : d_i^{(s)}(x) \neq 0\}\right|}{L}, \quad 1 \leq i_0 < k_s - L,\ x \in X_s, \tag{8.5}$$

(3) with $k_s = m_s l^2 + r_s$, $0 \leq r_s < l^2$, one has

$$m_s l > 2K, \quad \frac{K}{m_s} < \frac{1}{2}, \quad \text{and} \quad \frac{4r_s}{m_s l - 2K} < \frac{\delta}{2}. \tag{8.6}$$

Then, for each $s = 1, ..., S$, consider $\mathrm{M}_{k_s}(\mathbb{C}) \subseteq \mathrm{M}_{k_s}(\mathrm{C}(X_s))$, and consider

$$p_i^{(s)} := \mathrm{diag}\{\overbrace{\underbrace{0_{m_s}, ..., 0_{m_s}, 1_{m_s}}_{im_s}, 0_{m_s}, ..., 0_{m_s}}^{l^2 m_s}, 0_{r_s}\}, \quad i = 1, ..., l^2.$$

Note that $p_1^{(s)}, p_2^{(s)}, ..., p_{l^2}^{(s)} \subseteq \mathrm{M}_{k_s}(\mathbb{C})$ have the same rank and are mutually orthogonal. Therefore, there is a homomorphism

$$\phi_s : \mathrm{M}_{l^2}(\mathbb{C}) \ni e_{i,i} \mapsto p_i^{(s)} \in \mathrm{M}_{k_s}(\mathbb{C}) \subseteq A_2.$$

Define

$$\phi := \bigoplus_s \phi_s : \mathrm{M}_{l^2}(\mathbb{C}) \to \bigoplus_s \mathrm{M}_{k_s}(\mathbb{C}) \subseteq A_2,$$

and set $e_i = \phi(e_{i,i})$, $i = 1, 2, ..., l^2$, $s_k = e_{l(k-1)+1} + \cdots + e_{l(k-1)+4}$, $E_k = e_{l(k-1)+1} + \cdots + e_{lk}$, $k = 1, ..., l$, and $h = \phi(1)$.

Then, since $m_s l > 2K$, by (8.3),

$$E_{k_1} a' E_{k_2} = 0, \quad k_2 - k_1 \geq 2.$$

For each $k = 1, 2, ..., l$, consider

$$c_k := \bigoplus_s \mathrm{diag}\{\overbrace{\underbrace{0_{m_s}, ..., 0_{m_s}}_{l(k-1)m_s}, \underbrace{\underbrace{0_{m_s}, ..., 0_{m_s}}_{4m_s}, c^{(s)}_{(l(k-1)+4)m_s+1}, ..., c^{(s)}_{lkm_s}}_{lm_s}, 0_{m_s}, ..., 0_{m_s}}^{l^2 m_s}, 0_{r_s}\},$$

and

$$d_k := \bigoplus_s \mathrm{diag}\{\overbrace{\underbrace{0_{m_s}, ..., 0_{m_s}}_{l(k-1)m_s}, \underbrace{d^{(s)}_{l(k-1)m_s+1}, ..., d^{(s)}_{lkm_s}}_{lm_s}, 0_{m_s}, ..., 0_{m_s}}^{l^2 m_s}, 0_{r_s}\}.$$

Then it is clear that $c_k \perp s_k$, $c_k \perp d_k$, $c_k E_k = c_k$ and $d_k E_k = d_k$.

Note that, by (8.4), (8.5), and (8.6),

$$\frac{1}{4}\mathrm{rank}(c_k(x)) > \frac{1}{4} \cdot \frac{\delta}{2}((l-4)m_s - 2K) > 3m_s = \mathrm{rank}(s_k(x))$$

and

$$\frac{1}{4}\mathrm{rank}(d_k(x)) > \frac{1}{4} \cdot \frac{\delta}{2}(lm_s - 2K) > r_s = \mathrm{rank}((1-h)(x)).$$

Since s_k, c_k, d_k and $1 - h$ are diagonal elements, by Theorem 7.8 of [17], one has that

$$s_k \precsim c_k \quad \text{and} \quad 1 - h \precsim d_k.$$

Therefore, a' is a $\mathcal{D}_0$-operator (with $p = q = l$, $l = 2$ and $r = 4$ in Definition 5.1).

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